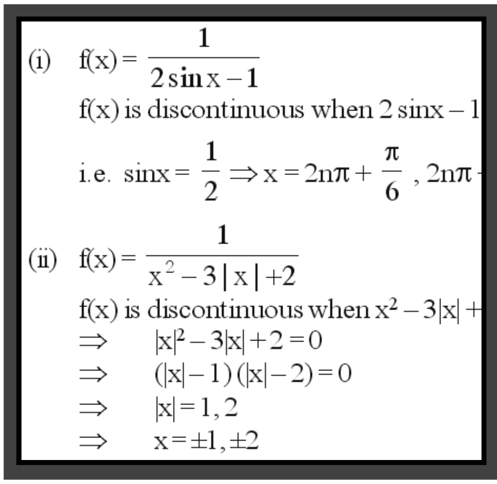


CHAPTER

2

CONTINUITY OF FUNCTIONS

2.1 DEFINITION OF CONTINUITY

Earlier we came across continuous functions and widely used their properties when constructing the graphs of simple functions, though the term "continuous function" was not used that time since the definition of this notion was not given then.

In the first stage the graphs of functions, say, $y = ax + b$, $y = ax^2$ or $y = ax^3$ were plotted point by point. The procedure was as follows: we first tabulated the values of a given function for certain values of the argument and then we constructed the points whose coordinates were put down in the table and then joined the plotted points with a "continuous curve". Thus, we obtained the graph of the given functions. We did not notice then that the graph of such functions were continuous.

The geometrical concept of continuity for a function which possesses a graph is that the function is continuous if its graph is an unbroken curve. A point at which there is a sudden break in the curve is thus a point of discontinuity.

The notion of continuity is a direct consequence of the concept of limit. The special class of functions known as continuous functions possesses many important properties which will be investigated in this chapter.

Continuity at a point

The question for our consideration is as follows: given any function f(x) defined in the neighbourhood of a, is the function f(x) continuous or not at x = a?

The graph of y = f(x) is said to be continuous at x = a if it can be traced with a continuous motion – without any jump – of the pen from left to right at x = a.

Definition A function f is continuous at x = a if the following three conditions are met:

(i) f(x) is defined at x = a.

(ii) $\lim_{x \to a} f(x)$ exists.

(iii) $\lim_{x \to a} f(x) = f(a)$.

In other words, function f(x) is said to be continuous at $x = a$, if $\lim_{x \to a} f(x) = f(a)$.

In terms of one-sided limits, f(x) is continuous at x = a if L.H.L. = f(a) = R.H.L.

i.e. $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

i.e. $\lim_{h \to 0} f(a - h) = f(a) = \lim_{h \to 0} f(a + h)$.

There is another way to discuss about continuity. Let there be a function y = f(x), x ∈ (a, b), and let x_0 be a certain value of the argument from the interval (a, b). Then, if x ∈ (a, b) is another value of the argument, the difference $x - x_0$ is called the increment of the argument and is denoted by Δx, and the difference

$$f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

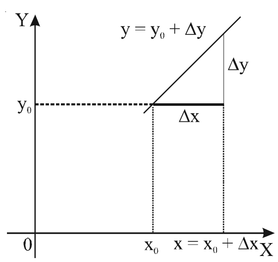
is termed as the increment of the function f at the point x_0 and is denoted as Δf or Δy.

If the function f is continuous at the point x_0, then, by definition, $\lim_{x \to x_0} f(x) = f(x_0)$ and

consequently, $\lim_{x \to x_0} f(x) - f(x_0) = 0$,

which means $\lim_{\Delta x \to 0} \Delta f = 0$.

It follows from the last relation that if f(x) is continuous at the point x_0, then to a small increment of the argument there corresponds a small increment of the function or, the increment of the function f is an infinitely small quantity as $\Delta x \to 0$.



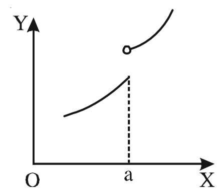
Formally, the function f(x) is continuous when x = a, if given ε, a number δ can be found, such that, whenever $|x - a| \le \delta$, we have $|f(x) - f(a)| < \varepsilon$.

Points of Discontinuity

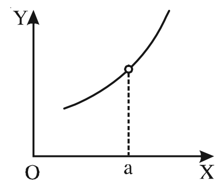
If a function f(x) is continuous at a point x = a, then the point a is called the point of continuity of the function f(x). Otherwise, when the limit of the function f(x) at the point a does not exist, or exists but is not equal to f(a), the function is said to be discontinuous at the point x = a, the latter being called the point of discontinuity of the function f(x).

In particular, if f(x) is defined for all points of the interval $(a - \delta, a + \delta)$ except for the point a, then x = a is also a point of discontinuity of the function f(x) in the interval. It follows from the aforegoing that a function is discontinuous at a given point if either (i) the given point fails to be in the domain of the function, or (ii) the function fails to be have a limit at the given point, or (iii) the limit of the function is unequal to the function's value at the given point.

(i)



(ii)



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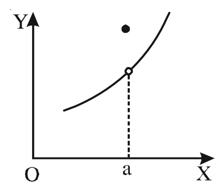
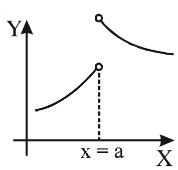
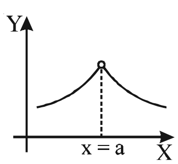


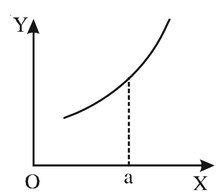
(iv)



(v)



(vi)



The function shown in figures (i) to (v) are discontinuous at x = a while that in (vi) is continuous at x = a. Thus, a function f can be discontinuous due to any of the following three reasons:

(i) $\lim_{x \to a} f(x)$ does not exist

i.e. $\lim_{x \to a^-} f(x) \ne \lim_{x \to a^+} f(x)$ [figures (i) and (iv)]

(ii) f(x) is not defined at x = c [figures (ii) and (v)]

(iii) $\lim_{x \to a} f(x) \ne f(c)$ [figure (iii)]

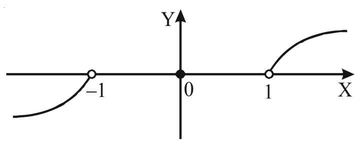
Geometrically, the graph of the function will exhibit a break at x= c .

Note: It should be noted that continuity of a function is the property of interval and is meaningful at x = a only if the function has a graph in the immediate neighbourhood of x = a.

For example, the discussion of continuity of

$f(x) = \frac{1}{x-1}$ at x = 1 is meaningful, but continuity of $f(x) = \ln x$ at x = –2 is meaningless.

Similarly, if f(x) has a graph as shown in the figure below, then continuity at x = 0 is meaningless since we can't approach 0 from either side of the point. Such a point is called an isolated point.



Consider the following examples:

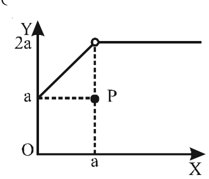
(i) The function $f(x) = \frac{1}{(1-x)^2}$ is discontinuous at x = 1. This function is not defined at the point x = 1.

(ii) The function $f(x) = \frac{\sin x}{|x|}$ has a discontinuity at

$x = 0$, since $f(0^+) = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$ and

$f(0^-) = \lim_{x\to 0^-} \frac{\sin x}{-x} = -1$.

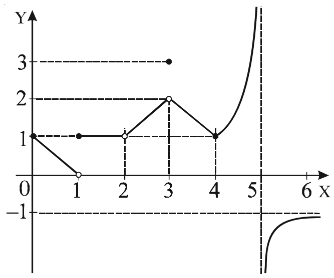
(iii) Let $f(x) = \begin{cases} \frac{x^2-a^2}{x-a} & \text{when } 0 \le x < a, \\ a & \text{when } x = a, \\ 2a & \text{when } x > a. \end{cases}$



Here $\lim_{x\to a^-} f(x) = 2a$, $f(a) = a$, $\lim_{x\to a^+} f(x) = 2a$,

and there is a discontinuity at x = a due to the isolated point P.

To understand explicitly the reasons of discontinuity, consider the graph of the following function y = f(x).



Let us comment on the continuity of the function.

(i) f is continuous at x = 0 and x = 4

(ii) f is discontinuous at x = 1 as limit does not exist

(iii) f is discontinuous at x = 2 as f (2) is not defined although the limit exists.

(iv) f is discontinuous at x = 3 as $\lim_{x\to 3} f(x) \ne f(3)$

(v) f is discontinuous at x = 5 as neither the limit exist nor f is defined at x = 5.

Example 1. Discuss the continuity of the function [cos x] at $x = \frac{\pi}{2}$, where [.] denotes the greatest integer function.

Solution L.H.L = $\lim_{x\to \frac{\pi}{2}^-} [\cos x] = 0$.

R.H.L = $\lim_{x\to \frac{\pi}{2}^+} [\cos x] = -1$.

$f\left(\frac{\pi}{2}\right) = \left[\cos\frac{\pi}{2}\right] = 0.$

Since, L.H.L ≠ R.H.L the limit does not exist.

So, the function is discontinuous at $x = \frac{\pi}{2}$.

Example 2. Test the continuity of the function f(x) at x = 0, where $f(x) = \frac{e^{1/x}}{1+e^{1/x}}$, when x ≠ 0 and f(0) = 0.

Solution Given, f(0) = 0

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{1/x}}{1+e^{1/x}} = \frac{0}{1+0} = 0$$

$$\text{since } \lim_{x\to 0^-}\left(\frac{1}{x}\right) = -\infty$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{1/x}}{1+e^{1/x}}$$

$$\lim_{x\to 0^+} \frac{1}{\frac{1}{e^{1/x}}+1} = \frac{1}{0+1} = 1$$

since $\lim_{x\to 0^+}\left(\frac{1}{x}\right)=\infty$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$,

f(x) is discontinuous at x = 0.

Example 3. Find whether f(x) is continuous or not at x = 1, if $f(x)=\begin{cases}\sin\frac{\pi x}{2}, & x<1\\ [x], & x\ge 1\end{cases}$,

where [.] denotes the greatest integer function.

Solution For continuity at x = 1, we determine, f(1), $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

Now, f(1) = [1] = 1.

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\sin\frac{\pi x}{2}=\sin\frac{\pi}{2}=1.$$

and $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}[x]=1.$

So $f(1)=\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x).$

∴ f(x) is continuous at x = 1.

One-sided Continuity

A function f defined in some neighbourhood of a point a for $x \le a$ is said to be continuous at a from the left if $\lim_{x\to a^-} f(x)=f(a)$.

A function f defined in some neighbourhood of a point a for $x \ge a$ is said to be continuous at a from the right if $\lim_{x\to a^+} f(x)=f(a)$.

One-sided continuity is a collective term for functions continuous from the left or from the right.

If the function f is continuous at a, then it is continuous at a from the left and from the right . Conversely, if the function f is continuous at a from the left and from the right, then $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} f(x)=f(a)$.

The function y = sgn x is neither left continuous nor right continuous at x = 0.

The function $y=\sin^{-1} x$ is left continuous at x = 1. We cannot discuss the right continuity here as the function is not defined in the right neighbourhood of x = 1.

The function y = {x} is right continuous at x = 0 but left discontinuous there.

At each integer n, the function f(x) = [x] is continuous from the right but discontinuous from the left because

$$\lim_{x\to n^+} f(x)=\lim_{x\to n^+}[x]=n=f(n)$$

but $\lim_{x\to n^-} f(x)=\lim_{x\to n^-}[x]=n-1\neq f(n)$.

Example 4. Let $f(x)=\begin{cases}x^2 & \text{if } x>2\\ x+1 & \text{if } x\le 2\end{cases}$

Show that f is continuous from the left at 2, but not from the right.

Solution f(2) = 2 + 1 = 3

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(x+1)=3$ and

$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} x^2=4$

Since $\lim_{x\to 2^-} f(x)=f(2)$, f is continuous from the left at 2 and $\lim_{x\to 2^+} f(x)\neq f(2)$, f is not continuous from the right at 2.

Continuity at End Points

Let a function y = f(x) be defined on [a, b].

Then the function f(x) is said to be continuous at the left end point x = a if, $f(a)=\lim_{x\to a^+} f(x)$,

and f(x) is said to be continuous at the right end point x = b if, $f(b)=\lim_{x\to b^-} f(x)$.

For example, consider the function f(x) = {x}, $0\le x\le 1$. It is continuous at x = 0 and discontinuous at x = 1.

Example 5. Discuss the continuity of $f(x)=\sqrt{\frac{x-a}{b-x}}$ where a < b, at x = a and x = b.

Solution We notice that the domain of the function is (a, b). At the left end point x = a, we have $\lim_{x\to a^+} f(x)=0=f(a)$. Hence f is continuous at x = a.

At the right end point x = b, f(b) is undefined and $\lim_{x\to b^-} f(x)=\infty$. Hence f is discontinuous at x = b.

Continuous Extension to a Point

A function may have a limit at a point where it is undefined. Then we can extend the definition of the function at that point to make it continuous there. Suppose f(a) is not defined, but $\lim_{x\to a} f(x)=L$ exists. If the point a is added to the domain of definition of the function f(x) and if at that new point the value of the function is put equal to the common value of the left hand and right hand limits, the (new) function F(x) thus obtained, is continuous at the point a. Then, we can define the new function F(x) by the formula

$$F(x)=\begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x=a \end{cases}$$

The function F is continuous at x = a. It is called the continuous extension of f to x = a.

For example, the function $y=\frac{\sin x}{x}$ is not defined at the point x = 0. But since $\lim_{x\to 0}\frac{\sin x}{x}=1$ we can introduce a new function, defined for all the values of x and coinciding with the old one for $x \neq 0$, which is everywhere continuous:

$$F(x)=\begin{cases} \frac{\sin x}{x} & \text{for } x\neq 0 \\ 1 & \text{for } x=0 \end{cases}.$$

Example 6. Let $f(x)=\frac{\ln(1+ax)-\ln(1-bx)}{x}$.

Find the value which should be assigned to f at x = 0, so that it is continuous at x = 0.

Solution $\lim_{x\to 0} f(x)$

$$=\lim_{x\to 0} a\left[\frac{\ln\ (1+ax)}{ax}\right]+b\left[\frac{\ln\ (1-bx)}{(-bx)}\right]$$

$$= a.1+b.1=a+b.$$

$$\left[\because \lim_{x\to 0}\frac{\log(1+x)}{x}=1\right]$$

So, f(0) = a + b, if f is continuous at x = 0.

Example 7. A function y = f (x) is defined as

$$f(x)=\begin{cases} k\sin\frac{(x+3)\pi}{6} & \text{for } x\le 2 \\ \frac{3-\sqrt{11-x}}{x-2} & \text{for } x>2 \end{cases}$$

If f (x) is continuous at x = 2, then find the value of k.

Solution Since f (x) is continuous at x = 2

$$\lim_{x\to 2} f(x)=f(2) \Rightarrow \lim_{x\to 2} f(x)=\frac{k}{2}.$$

Now, $f(2^-)=k\sin\left(\frac{5\pi}{6}\right)=\frac{k}{2}$ and

$$f(2^+)=\lim_{x\to 2}\frac{3-\sqrt{11-x}}{x-2}$$

$$=\lim_{x\to 2}\frac{(x-2)}{(x-2)\left(3+\sqrt{11-x}\right)}=\frac{1}{6}.$$

$$\therefore\ \frac{k}{2}=\frac{1}{6} \Rightarrow k=\frac{1}{3}.$$

Example 8. Let $f(x)=\frac{\ln(1+x)^{1+x}-x}{x^2}$, then find the value of f(0) so that the function f is continuous at x = 0.

Solution $\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{(1+x)\ln(1+x)-x}{x^2}\left(\frac{0}{0}\text{form}\right)$

$$=\lim_{x\to 0}\frac{\ln(1+x)+1-1}{2x} \quad \text{(by L'Hospital's Rule)}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{\ln(1+x)}{x}=\frac{1}{2}.$$

For continuity, we must have $f(0)=\lim_{x\to 0} f(x)$.

Hence, $f(0)=\frac{1}{2}$.

Example 9. Examine the continuity of the function

$$f(x)=\begin{cases} \frac{x^2-1}{x^2-2|x-1|-1}, & x\neq 1 \\ \frac{1}{2}, & x=1 \end{cases}, \text{at } x=1.$$

Solution $\text{L.H.L}=f(1^+)=\lim_{x\to 1^+}\frac{x^2-1}{x^2-2|x-1|-1}$

$$=\lim_{x\to 1^+}\frac{x^2-1}{x^2-2(x-1)-1}$$

$$=\lim_{x\to 1^+}\frac{(x+1)}{(x+1)-2}=\infty$$

There is no need to find the R.H.L. at x = 1 since L.H.L. is non-existent.

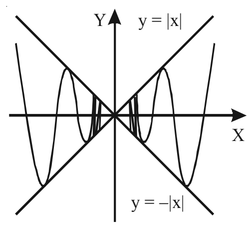
Thus, f(x) is discontinuous at x = 1.

Example 10. Check the continuity of the function $f(x)=\begin{cases} x\sin(1/x), & x\neq 0 \\ 0, & x=0 \end{cases}$ at $x=0$.

Solution Consider the limit $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

If $x > 0, -x \leq x \sin(1/x) \leq x$,
and if $x < 0, x \leq x \sin(1/x) \leq -x$.
Thus, for $x \neq 0, -|x| \leq x \sin(1/x) \leq |x|$.

Since both $|x| \longrightarrow 0$ and $-|x| \longrightarrow 0$ as $x \longrightarrow 0$, the Sandwich theorem applies and we can conclude that $x \sin(1/x) \longrightarrow 0$ as $x \longrightarrow 0$. This is illustrated in the following figure.



$y = x \sin\left(\frac{1}{x}\right)$

It follows that the function is continuous at $x = 0$, since the value of the function and the value of the limit are the same at 0. This shows that the behaviour of a function can be very complex in the vincinity of $x = a$, even though the function is continuous at a.

Example 11. Test the continuity of $f(x)$ at $x = 0$ if

$$f(x) = \begin{cases} (x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}.$$

Solution L.H.L. $= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$$= \lim_{h \to 0} (0-h+1)^{2-\left(\frac{1}{|0-h|}+\frac{1}{(0-h)}\right)}$$

$$= \lim_{h \to 0}(0-h+1)^{2-\left(\frac{1}{|0-h|}+\frac{1}{(0-h)}\right)}$$

$$= \lim_{h \to 0} (1-h)^2 = (1-0)^2 = 1.$$

R.H.L $= \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$

$$= \lim_{h \to 0} (h+1)^{2\left(\frac{1}{|h|}+\frac{1}{h}\right)}$$

$$= \lim_{h \to 0} (h+1)^{2-\frac{2}{h}}$$

$= (1+0)^{2-\infty} = 1^{-\infty} = 1.$

$f(0) = 0.$

$\therefore$ L.H.L. = R.H.L. $\neq f(0)$

Hence, $f(x)$ is discontinuous at $x = 0$.

Example 12. Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{a^{2[x]+\{x\}}-1}{2[x]+\{x\}} & , \quad x \neq 0 \\ \log_e a & , \quad x = 0 \end{cases} \quad (a \neq 1)$$

at $x = 0$, where $[x]$ and $\{x\}$ are the greatest integer part and fractional part of x respectively.

Solution $f(0) = \log_e a$

L.H.L. $= \lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$

$$= \lim_{h \to 0} \frac{a^{2[0-h]+\{0-h\}}-1}{2[0-h]+\{0-h\}}$$

$$= \lim_{h \to 0} \frac{a^{2[0-h]+\{-1+(1-h)\}}-1}{2[0-h]+\{-1+(1-h)\}}$$

$$= \lim_{h \to 0} \frac{a^{-2+(1-h])}-1}{-2(1-h)} = \frac{a^{-1}-1}{-1} = 1-\frac{1}{a}.$$

R.H.L. $= \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$

$$= \lim_{h \to 0} \frac{a^{2[-)]+\{0+h\}-1}}{2[0+h]+\{0+h\}}$$

$$= \lim_{h \to 0} \frac{a^{0+h}-1}{0+h}$$

$$= \lim_{h \to 0} \frac{a^h-1}{h} = \log_e a$$

We find that $1 - 1/a = \log_e a$ only when $a = 1$, which is not acceptable. Since L.H.L. ≠ R.H.L. = f(0), f(x) is discontinuous at $x = 0$.

Example 13. If the function

$f(x) = \dfrac{\tan(\tan x) - \sin(\sin x)}{\tan x - \sin x}$ $(x \neq 0)$ is continuous at $x = 0$, then find the value of $f(0)$.

Solution $f(0) = \lim_{x \to 0} \dfrac{\tan(\tan x) - \sin(\sin x)}{\tan x - \sin x}$

$$= \lim_{x \to 0} \frac{\tan(\tan x) - \sin(\sin x)}{\frac{\tan x}{x}\left(\frac{1-\cos x}{x^2}\right)x^3}$$

$$= 2\lim_{x \to 0} \frac{\tan(\tan x) - \sin(\sin x)}{x^3}$$

$$= 2\lim_{x \to 0} \frac{1}{x^3}\left(\left(\tan x + \frac{\tan^3 x}{3} + \frac{2}{15}\tan^5 x + \ldots\ldots\right)\right.$$

$$-\left(\sin x-\frac{\sin^3 x}{3!}+\frac{\sin^5 x}{5!}-.........\right)\Bigg)$$

$$=2\lim_{x\to 0}\left(\left(\frac{\tan x-\sin x}{x^3}\right)+\frac{\left(\frac{\tan^3 x}{3}+\frac{\sin^3 x}{3!}\right)}{x^3}+....\right)$$

$$=2\lim_{x\to 0}\left(\left(\frac{\tan x}{x}\right)\left(\frac{1-\cos x}{x^2}\right)+\frac{1}{3}+\frac{1}{6}\right)$$

$$=2\left[\frac{1}{2}+\frac{1}{2}\right]=2.$$

Example 14. Let $f(x)=\dfrac{\sqrt{2}\cos x-1}{\cot x-1}$ $\forall\ x\in\left(0,\dfrac{\pi}{2}\right)$ except at $x=\dfrac{\pi}{4}$. Define $f\left(\dfrac{\pi}{4}\right)$ so that f(x) may be continuous at $x=\dfrac{\pi}{4}$.

Solution f(x) will be continuous at $x=\dfrac{\pi}{4}$, if $\lim_{x\to\pi/4} f(x)=f\left(\dfrac{\pi}{4}\right)$

$$\therefore\quad f\left(\frac{\pi}{4}\right)=\lim_{x\to\pi/4}\frac{\sqrt{2}\cos x-1}{\cot x-1}$$

$$=\lim_{x\to\pi/4}\frac{(\sqrt{2}\cos x-1)\sin x}{\cos x-\sin x}$$

$$=\lim_{x\to\pi/4}\frac{(\sqrt{2}\cos x-1)(\sqrt{2}\cos x+1)(\cos x+\sin x)\sin x}{(\sqrt{2}\cos x+1)(\cos x-\sin x)(\cos x+\sin x)}$$

$$=\lim_{x\to\pi/4}\frac{(2\cos^2 x-1)}{(\cos^2 x-\sin^2 x)}\frac{(\cos x+\sin x)\sin x}{(\sqrt{2}\cos x+1)}$$

$$=\lim_{x\to\pi/4}\frac{\sin x(\cos x+\sin x)}{\sqrt{2}\cos x+1}$$

$$=\frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)}{\sqrt{2}.\frac{1}{\sqrt{2}}+1}=\frac{1}{2}.$$

Example 15. Let $f(x)=\dfrac{e^{\tan x}-e^x+ln(\sec x+\tan x)-x}{\tan x-x}$ be a continuous function at $x=0$. Find the value of $f(0)$.

Solution For continuity of f at $x=0$, we have

$$f(0)=\lim_{x\to 0} f(x)$$

$$=\lim_{x\to 0}\frac{e^{\tan x}-e^x}{\tan x-x}+\lim_{x\to 0}\frac{\ln(\sec x+\tan x)-x}{\left(\frac{\tan x-x}{x^3}\right)x^3}$$

$$=\lim_{x\to 0}\frac{e^x(e^{\tan x-x}-1)}{\tan x-x}+3\lim_{x\to 0}\frac{\ln(\sec x-\tan x)-x}{x^3}$$

$$=1+3\lim_{x\to 0}\frac{\sec x-1}{3x^2}\quad\text{(by L'Hospital's Rule)}$$

$$=1+\frac{1}{2}=\frac{3}{2}.$$

Example 16. Let a function $f(x)$ be defined in the neighborhood of as

$$f(x)=\begin{cases}\dfrac{\ln(2-\cos 2x)}{\ln^2(1+\sin 3x)} & \text{for } x<0\\[2ex] \dfrac{e^{\sin 2x}-1}{\ln(1+\tan 9x)} & \text{for } x>0\end{cases}$$

Find whether it is possible to define $f(0)$ so that f may be continuous at $x=0$.

Solution $\lim_{h\to 0} f(0+h)$

$$=\lim_{h\to 0}\frac{e^{\sin 2h}-1}{\ln(1+\tan 9h)}\frac{\sin 2h.(2h)}{\sin 2h.(2h)}$$

$$=\lim_{h\to 0}\frac{e^{\sin 2h}-1}{\sin 2h}.\lim_{h\to 0}\frac{\sin 2h}{2h}\lim_{h\to 0}\frac{2h}{\ln(1+\tan 9h)}$$

$$=1.1.\lim_{h\to 0}\frac{2h}{\tan 9h\ \ln(1+\tan 9h)\cot 9h}$$

$$=\lim_{h\to 0}\frac{2h}{\tan 9h}=\frac{2}{9}.\quad \lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0}\frac{\ln(1+2\sin^2 h)}{2\sin^2 h}.\lim_{h\to 0}\frac{2\sin^2 h}{\ln^2(1-\sin 3h)}$$

$$=\lim_{h\to 0}\frac{2\sin^2 h}{h^2}.\lim_{h\to 0}\frac{h^2}{\ln^2(1-\sin 3h)}$$

$$=\lim_{h\to 0}\frac{1}{9h^2}\ln(2-\cos 2h)$$

$$= \frac{1}{9}\lim_{h\to 0} \ln(2-\cos 2h)^{1/h^2}$$

$$= \lim_{h\to 0} \frac{2\cdot h^2(-\sin 3h)^2}{\sin^3 3h\cdot \ln^2(1-\sin 3h)} = \frac{2}{9}$$

Since $f(0^+) = f(0^-) = \frac{2}{9}$ the limit exists.

Therefore, it is possible to define f(0) such that f is continuous at x = 0.

$$\therefore \quad f(0) = \frac{2}{9}.$$

Example 17. Determine the value of p, so that the function
$$f(x) = \begin{cases} \dfrac{x^2+2\cos x-2}{x^4} & \text{for } x<0 \\ p & \text{for } x=0 \\ \dfrac{\sin x - \ell n(e^x\cos x)}{6x^2} & \text{for } x>0 \end{cases}$$
is continuous at x = 0.

Solution $\lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{h^2-2(1-\cos h)}{h^4}$

$$= \lim_{h\to 0}\frac{h^2-4\sin^2\frac{h}{2}}{h^4}$$

$$= \lim_{h\to 0}\frac{h+2\sin\frac{h}{2}}{h}\cdot\lim_{h\to 0}\frac{h-2\sin\frac{h}{2}}{h^3}$$

$$= 2\cdot\lim_{t\to 0}\frac{2t-2\sin t}{8t^3} \quad \text{where } h = 2t$$

$$= \lim_{t\to 0}\frac{t-\sin t}{2t^3} = \frac{1}{12}.$$

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\sin h - \ln(e^h\cos h)}{6h^2}$$

$$= \lim_{h\to 0}\frac{\sin h - h - \ln\cos h}{6h^2}$$

$$= \frac{\sinh - h}{6h^2} - \frac{1}{6}\ln(\cosh)^{\frac{1}{h^2}}$$

$$= \lim_{h\to 0}\frac{\sin h - h}{h^3}\cdot h + \lim_{h\to 0} -\frac{1}{6}\ln(\cos h)^{\frac{1}{h^2}}$$

$$= 0 - \frac{1}{6}\ln e^{\lim\limits_{h\to 0}\frac{1}{h^2}(\cos h-1)} = \frac{1}{12}$$

Hence, $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0) = \frac{1}{12} = p.$

Example 18. If $f(x) = \frac{\sin 2x + A\sin x + B\cos x}{x^3}$ is continuous at x = 0, find the values of A and B. Also find f(0).

Solution As f(x) is continuous at x = 0,

$$f(0) = \lim_{x\to 0}\frac{\sin 2x + A\sin x + B\cos x}{x^3}.$$

As denominator $\to 0$, when $x \to 0$, numerator should also approach 0, which is possible only if

$\sin 2(0) + A\sin(0) + B\cos(0) = 0 \Rightarrow B = 0$

$$\therefore \quad f(0) = \lim_{x\to 0}\frac{\sin 2x + A\sin x}{x^3}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{2\cos x + A}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\cos x + A}{x^2}\right)$$

Again we can see that denominator $\to 0$ as $x \to 0$.

$\therefore$ Numerator should also approach 0 as $x \to 0$

$\Rightarrow 2 + A = 0 \Rightarrow A = -2.$

$$\Rightarrow f(0) = \lim_{x\to 0}\left(\frac{2\cos x - 2}{x^2}\right) = \lim_{x\to 0}\left(\frac{-4\sin^2 x/2}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\sin^2 x/2}{x^2/4}\right) = -1$$

So, we get A = – 2, B = 0 and f(0) = – 1.

Example 19. Let

$$f(x) = \begin{cases} \dfrac{a(1-x\sin x)+b\cos x+5}{x^2} & , x<0 \\ 3 & , x=0 \\ \left\{1+\left(\dfrac{cx+dx^3}{x^2}\right)\right\}^{1/x} & , x>0 \end{cases}$$

If f is continuous at x = 0, then find the values of a, b, c and d.

Solution f(0) = 3

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0}\left\{1+\left(\frac{ch+dh^3}{h^2}\right)\right\}^{1/h}$$

Since f is continuous at x = 0, R.H.L. exists.
For existence of R.H.L., c must be 0.

$\therefore$ R.H.L. $= \lim_{h\to 0} \{1+dh\}^{1/h}$ (form 1^∞)

$= e^{\lim_{h\to 0}(1+dh-1)} = e^d$

L.H.L. $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(0-h)$

$= \lim_{h\to 0} \dfrac{a(1-(-h)\sin(-h)) + b\cos(-h)+5}{(-h)^2}$

$= \lim_{h\to 0} \dfrac{a(1-h\sin h) + b\cos h + 5}{h^2}$

For finite value of L.H.L. the numerator must tend to 0 as $h \to 0$.

$\therefore$ $a + b + 5 = 0.$

L.H.L. $= \lim_{h\to 0} \dfrac{a - ah\sin h - (5+a)\cos h + 5}{h^2}$

$= \lim_{h\to 0} \dfrac{(5+a)(1-\cos h) - a\, h\sin h}{h^2}$

$= \lim_{h\to 0} \left\{\dfrac{(a+5)(1-\cos h)(1+\cos h)}{h^2(1+\cos h)} - \dfrac{a\, h\sin h}{h^2}\right\}$

$= \lim_{h\to 0} \left\{\dfrac{(a+5)\sin^2 h}{h^2(1+\cos h)} - \dfrac{a\sin h}{h}\right\}$

$= \dfrac{(a+5)(1)^2}{1+1} - a\,.\,1$

$= \dfrac{a+5}{2} - a = \left(\dfrac{5-a}{2}\right).$

Now, f(x) is continuous at x = 0.

$\therefore$ L.H.L. = R.H.L. = f(0)

$\dfrac{5-a}{2} = e^d = 3$

$\therefore$ $a = 1, d = \ln 3, c = 0, b = -6.$

Example 20. Let f(x)

$$= \begin{cases} \dfrac{1 + a\cos 2x + b\cos 4x}{x^2 \sin^2 x} & \text{if } x \neq 0 \\ c & \text{if } x = 0 \end{cases},$$

be continuous at x = 0, then find the values of a, b and c.

Solution $\lim_{x\to 0} \dfrac{1 + a\cos 2x + b\cos 4x}{x^4}$

As $x \to 0$, Denominator $\to 0$ and Numerator $\to 1 + a + b$

For existence of limit, $a + b + 1 = 0$ ...(1)

$\therefore$ $c = \lim_{x\to 0} \dfrac{a\cos 2x + b\cos 4x - (a+b)}{x^4}$ ...(2)

$= -\lim_{x\to 0} \dfrac{\dfrac{a(1-\cos 2x)}{x^2} + \dfrac{b(1-\cos 4x)}{x^2}}{x^2}$

The limit of numerator $= 4a\left(\dfrac{1}{2}\right) + 16b\left(\dfrac{1}{2}\right)$

$\Rightarrow$ $2a + 8b = 0$ $\Rightarrow$ $a = -4b$ ...(3)

From (1) and (2) $-4b + b = -1$

$\Rightarrow$ $b = \dfrac{1}{3}$ and $a = -\dfrac{4}{3}$

From (2), $c = \lim_{x\to 0} \dfrac{4(1-\cos 2x) - (1-\cos 4x)}{3x^2}$

$= \lim_{x\to 0} \dfrac{8\sin^2 x - 2\sin^2 2x}{3x^4}$

$= \lim_{x\to 0} \dfrac{8\sin^2 x - 8\sin^2 x\cos^2 x}{3x^4}$

$= \lim_{x\to 0} \dfrac{8}{3} \cdot \dfrac{\sin^2 x}{x^2} \cdot \dfrac{\sin^2 x}{x^2} = \dfrac{8}{3}.$

Example 21. Discuss the continuity of the function, $f(x) = \lim_{n\to\infty} \dfrac{\ln(2+x) - x^{2n}\sin x}{1 + x^{2n}}$ at $x = 1$.

Solution We have $f(1) = (\ln 3 - \sin 1)/2$

and $\lim_{n\to\infty} x^{2n} = \begin{cases} 0, & \text{if } x^2 < 1 \\ \infty, & \text{if } x^2 > 1 \end{cases}$

$\therefore$ For $x^2 < 1$, we have

$f(x) = \lim_{n\to\infty} \dfrac{\ln(2+x) - x^{2n}\sin x}{1 + x^{2n}} = -\ln(2+x)$

Again for $x^2 > 1$, we have

$f(x) = \lim_{n\to\infty} \dfrac{\dfrac{1}{x^{2n}}\ln(2+x) - \sin x}{1 + \dfrac{1}{x^{2n}}} = -\sin(x)$

Here, as $x \to 1$

$\lim_{x\to 1^-} f(x) = \ln 3$ and $\lim_{x\to 1^+} f(x) = -\sin 1$

So, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x).$

Therefore, f(x) is discontinuous at x = 1.

Example 22. Find the value of f(1) if the function

$$f(x)=\lim_{n\to\infty}\frac{x^{m+1}-(m+1)x+m}{(x-1)^2},\ x\neq 1$$

is continuous at x = 1.

Solution We have

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{x(x^m-1)-m(x-1)}{(x-1)^2},$$

$$=\lim_{x\to 0}\frac{x(x-1)(1+x+x^2+....+x^{m-1})-m(x-1)}{(x-1)^2}$$

$$=\lim_{x\to 0}\frac{(x+x^2+x^3+....+x^m)-(1+1+...m\text{ times})}{x-1}$$

$$=\lim_{x\to 0}\frac{(x-1)+(x^2-1)+(x^3-1)+.....+(x^m-1)}{x-1}$$

$$=\lim_{x\to 0}\left[1+\frac{x^2-1}{x-1}+\frac{x^3-1}{x-1}+.....+\frac{x^m-1}{x-1}\right]$$

$$=1+2+3+....+m=\frac{m(m+1)}{2}$$

Hence, for f to be continuous at x = 1, we should have

$$f(1)=\frac{m(m+1)}{2}.$$

Example 23. A function f(x) satisfies f(x + y) = f(x) . f(y) for all x and y ∈ R. Show that the function is continuous for all values of x if it is continuous at x = 1.

Solution We have f(x + y) = f(x) . f(y) ...(1)

Putting x = 0, y = 0 in (1)

we have f(0) = f(0). f(0) ⇒ f(0) = 0 or 1.

If f(0) = 0 then putting y = 0 in (1)

we get f(x) = f(x) . f(0)

⇒ f(x) = 0 for all x.

Hence f is continuous for all x. Here the continuity of f at x = 1 is not required as a condition.

If f(0) = 1 then we have the following:

Putting x = 1, y = –1 in (1)

we have f(0) = f(1). f(–1) ⇒ f(1). f(–1) = 1

Hence f(1) is non-zero. ...(2)

As the function is continuous at x = 1,

we have $\lim_{h\to 0} f(1+h)=f(1)$

(using f(x + y) = f(x) . f(y))

⇒ $\lim_{h\to 0} f(1).f(h)=f(1)$

⇒ $\lim_{h\to 0} f(h)=1$ using (2) ...(3)

Now, we consider any arbitrary point x.

$$\lim_{h\to 0} f(x+h)=\lim_{h\to 0} f(x).f(h)$$

$$=f(x)\lim_{h\to 0} f(h)$$

$$=f(x) \text{ using (3)}$$

Hence, at any arbitrary point x, limit = function's value. Therefore, the function is continuous for all values of x.

Example 24. Let f be a function satisfying

$$f(x+y)+\sqrt{6-f(y)}=f(x)f(y) \text{ and } \lim_{x\to 0} f(x)=6.$$

Discuss the continuity of f.

Solution $\lim_{h\to 0} f(x+h)=\lim_{h\to 0}\left[f(x)f(h)-\sqrt{6-f(h)}\right]$

$$=f(x)\lim_{h\to 0} f(h)-\lim_{h\to 0}\sqrt{6-f(h)}$$

$$=f(x).6-0=6f(x)\neq f(x)$$

Putting x = 0, y = 0 in the given relation, we get

$f(0)+\sqrt{6-f(0)}=f^2(0).$ ⇒ f(0) ≠ 0

∴ f(x) = 0 for all x is not possible.

Thus, $\lim_{h\to 0} f(x+h)\neq f(x).$

Hence, f is discontinuous at all x.

Concept Problems A

1. Check the continuity of

$$f(t)=\begin{cases}\dfrac{\tan(\sin t)}{\sin t}, & \sin t\neq 0\\ 1, & \sin t=0\end{cases} \text{ at } t=0.$$

2. If $f(x)=\begin{cases}x+\lambda, & x<3\\ 4, & x=3\\ 3x-5, & x>3\end{cases}$ is continuous at x = 3 then find the value of λ.

3. If $f(x)=\dfrac{2-\sqrt{x+4}}{\sin(2x)}$, x ≠ 0 is continuous function at x = 0, then find the value of f(0).

4. If the function $f(x)=\begin{cases}\dfrac{x^2-(a+2)x+2a}{x-2}, & x\neq 2\\ 2, & x=2\end{cases}$ is continuous at x = 2 then find the value of a.

5. If f and g are continuous functions with f(3) = 5 and $\lim_{x\to 3}[2f(x)-g(x)]=4$, find g(3).

6. Use continuity to evaluate the limits :

(i) $\lim_{x\to 4}\frac{5+\sqrt{x}}{\sqrt{5+x}}$

(ii) $\lim_{x\to 1}\tan^{-1}\left(\frac{x^2-4}{3x^2-6x}\right)$

7. For the function $\phi(x) = x.l\text{n}\sin^2 x$ when x > 0, and $\phi(0) = 0$, discuss the right continuity at x = 0.

8. The function $f(x)=(x+1)^{\cot x}$ is undefined at x = 0. Find the value which should be assigned to f at x = 0, so that it is continuous at x = 0.

Practice Problems A

9. If possible find value of λ for which f(x) is continuous at $x=\frac{\pi}{2}$.

$$f(x) = \frac{1-\sin x}{1+\cos 2x}, \quad x<\frac{\pi}{2}$$
$$= \lambda, \quad x=\frac{\pi}{2}$$
$$= \frac{\sqrt{2x-\pi}}{\sqrt{4+\sqrt{2x-\pi}}-2}, \quad x>\frac{\pi}{2}$$

10. Find the value of f(0) so that the function $f(x)=\frac{\sqrt{1-x}-\sqrt[3]{1-x}}{x}$ is continuous at x = 0.

11. Show that the function f defined on R by setting $f(x)=|x|^m\sin\frac{1}{x}$, when $x\neq 0$, f(0) = 0, is continuous at x = 0 whenever m > 0.

12. The function $f(x)=1-x\sin\frac{1}{x}$ is meaningless for x = 0. How should one choose the value f(0) so that f(x) is continuous for x = 0?

13. Find the values of a and b such that the function

$$f(x) = x + a\sqrt{2}\sin x, \quad 0\le x<\frac{\pi}{4}$$
$$= 2x\cot x + b, \quad \frac{\pi}{4}\le x\le\frac{\pi}{2}$$
$$= a\cos 2x - b\sin x, \quad \frac{\pi}{2}<x\le\pi$$

is continuous at $\frac{\pi}{4}$ and $\frac{\pi}{2}$.

14. If $f(x)=(1+ax)^{\frac{1}{x}}$ x<0
$= b$ x=0
$= \frac{(x+c)^{\frac{1}{3}}-1}{x}$ x>0

find the values of a, b, c, f(x) is continuous at x = 0.

15. If $f(x)=\cos(x\cos\frac{1}{x})$ and $g(x)=\frac{l\text{n}(\sec^2 x)}{x\sin x}$ for $x\neq 0$ and they are both continuous at x = 0 then show that f (0) = g (0) = 1.

16. The function f(x) = a [x + 1] + b [x –1], where [.] is the greatest integer function then find the condition for which f(x) is continuous at x = 1.

17. Let f(x)

$$= \begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}} & \text{for } -\frac{\pi}{6}<x<0\\ b & \text{for } x=0\\ e^{\frac{\tan 2x}{\tan 3x}} & \text{for } 0<x<\frac{\pi}{6}\end{cases}$$

Find ‘a’ and ‘b’ if f is continuous at x = 0.

18. If f(x) is continuous in [0, 1] and $f\left(\frac{1}{2}\right)=1$ then find $\lim_{n\to\infty} f\left(\frac{\sqrt{n}}{2\sqrt{n+1}}\right)$.

19. Let $f(x)=\frac{\sin(e^{x-2}-1)}{\log(x-1)}$, $x\neq 2$.
If f(x) is continuous at x = 2 find f(2).

2.2 CONTINUITY IN AN INTERVAL

We can extend the concept of continuity and say that a function f(x) is continuous in an interval if it is continuous at every point in the interval.

Roughly speaking, a function is said to be continuous on an interval if its graph has no breaks, jumps, or holes in that interval. Continuity is important because, as we shall see, function with this property have many other desirable properties.

Continuity in an open interval

A function f is said to be continuous in an open interval (a , b) if f is continuous at each and every point lying in the interval (a , b) .

Continuity in a closed interval

A function f is said to be continuous in a closed interval [a, b] if :

(i) f is continuous in the open interval (a , b)

(ii) f is right continuous at 'a'

i.e. $\lim\limits_{x \to a^+} f(x) = f(a)$.

(iii) f is left continuous at 'b'

i.e. $\lim\limits_{x \to b^-} f(x) = f(b)$.

For instance, $f(x) = \sqrt{-x^2 + 3x - 2}$, where $1 \le x \le 2$, is a function continuous on this interval, since it is continuous at every point of the interval (1, 2), continuous on the right at the point x = 1, and continuous on the left at the point x = 2.

The function $y = \dfrac{1}{\sqrt{1-x^2}}$ is continuous in the open interval (–1, 1). It is discontinuous at the points x = – 1 and x = 1.

Similarly, f is continuous on the half-open interval (a, b] if it is continuous at each number between a and b and is continuous from the left at the endpoint b.

In the case where f is continuous on (–∞, ∞), we will say that f is continuous everywhere. The general procedure for showing that a function is continuous everywhere is to show that it is continuous at an arbitrary real number.

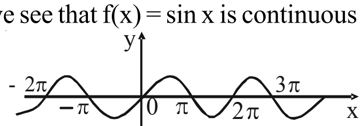
Continuity of elementary functions

All basic elementary functions are continuous in the intervals where they are defined.

The constant function and the identity function are continuous over R.

We know that y = sin x and y = cos x are continuous for every value of x.

From the graph we see that f(x) = sin x is continuous in its entire domain.



We see that the functions

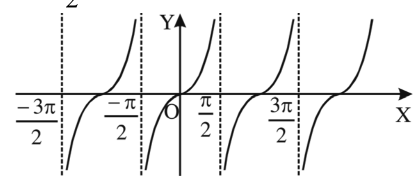
$$y = \tan x = \frac{\sin x}{\cos x},\ y = \cot x = \frac{\cos x}{\sin x}$$

$$y = \sec x = \frac{1}{\cos x},\ y = \text{cosec } x = \frac{1}{\sin x}$$

are continuous for all those values of x for which they are defined. The discontinuities of these four functions arise only when the denominators become zero and for such values of x, these functions themselves cease to be defined.

f(x) = tan x is continuous at all points except

$x = (2n + 1)\dfrac{\pi}{2}, n \in I$.



Thus, the domain of continuity of each of the functions sin x, cos x, tan x, cot x, sec x, and cosec x coincides with the corresponding domain of definition.

f(x) = tan x is continuous at all points except $x = (2n + 1)\dfrac{\pi}{2}, n \in I$.

Note: f is a continuous function if it is continuous at each number a in its domain.

The examination of the graph of the function y = 1/x in the vicinity of the point x = 0 clearly shows that it "splits" into two separate curves at that point.

However, the function f(x) = 1/x, whose domain consists of the intervals (–∞, 0) and (0, ∞) is continuous. Although this function explodes at 0, this does not prevent it from being a continuous function. The key to being continuous is that the function is continuous at each number in its domain. The number 0 is not in the domain of f(x) = 1/x.

However, f(x)=1/x is discontinuous in the interval (–∞, ∞)

Theorem Every elementary function is contin-uous at every point in its domain.

The polynomial function $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ is continuous over R .

It follows from the preceding theorem and the fact that constant function and the identity function are continuous over R.

If f(x) is continuous for any particular value of x, then any polynomial in f(x), such as

$a_0\{f(x)\}^n + a_1\{f(x)\}^{n-1} + ... + a_{n-1}\{f(x)\} + a_n$ is also continuous

The rational function $f(x) = \dfrac{a_0x^n + a_1x^{n-1} + ... + a_n}{b_0x^m + b_1x^{m-1} + ... + b_m}$ is continuous at every value of x except at those points where the denominator becomes zero.

For instance, the function $f(x) = \dfrac{3-x}{4x+7}$ is continuous throughout the entire number line except for the point $x = -\dfrac{7}{4}$, at which the denominator of the fraction vanishes. And the function $f(x) = -\dfrac{x^3+4x^2+x+1}{x^2+x+1}$ is continuous everywhere on R, since the denominator never vanishes.

Now, each of the irrational functions $\sqrt{(x-a)(b-x)}$, $\sqrt[3]{(x-a)(b-x)}$ and $\sqrt[3]{\left(\dfrac{x-a}{b-x}\right)}$ are continuous at each point in their domain, since these functions are elementary. All trigonometric functions, exponential and logarithmic functions are continuous in their domain.

Example 1. Test the following functions for continuity

(a) $f(x) = \dfrac{2x^5 - 8x^2 + 11}{x^4 + 4x^3 + 8x^2 + 8x + 4}$

(b) $f(x) = \dfrac{3\sin^3 x + \cos^2 x + 1}{4\cos x - 2}$

Solution

(a) A function representing a ratio of two continuous functions (polynomials in this case) is discontinuous only at points for which the denominator is zero.
But in this case $(x^4 + 4x^3 + 8x^2 + 8x + 4)$
$= (x^2 + 2x + 2)^2 = [(x+1)^2 + 1]^2 > 0$ (non-zero).
Hence f(x) is continuous everywhere.

(b) The function f(x) suffers discontinuities only at points for which the denominator becomes zero i.e. at the roots of the equation
$4\cos x - 2 = 0 \Rightarrow \cos x = 1/2.$
$\Rightarrow \quad x = 2n\pi \pm \pi/3, n \in I.$

Thus the function f(x) is continuous for all real x, except at the points $2n\pi \pm \pi/3$, $n \in I$.

Example 2. Let $f(x) = \dfrac{\sqrt{x^2+kx+1}}{x^2-k}$. Find all possible values of k for which f is continuous for every $x \in R$.

Solution $f(x) = \dfrac{\sqrt{x^2+kx+1}}{x^2-k}$

For f to be continuous $\forall\ x \in R$

$x^2 + kx + 1 \geq 0$ and $x^2 - k$ must not have any real root.

$\therefore \quad k^2 - 4 \leq 0 \quad$ and $k < 0$

$\Rightarrow \quad k \in [-2, 2] \quad$ and $k < 0$

From above, $k \in [-2, 0)$.

Example 3. Where is the function $F(x) = \ln(1 + \cos x)$ continuous?

Solution We know that f(x) = ln x is continuous and g(x) = 1 + cos x is continuous (because both y = 1 and y = cos x are continuous). Therefore, F(x) = f(g(x)) is continuous wherever it is defined.

Now ln(1 + cos x) is defined when 1 + cos x > 0. So it is undefined when cos x = – 1, and this happens when $x = \pm\pi, \pm 3\pi$, ... Thus, F has discontinuities when x is an odd multiple of π and is continuous on the intervals between these values.

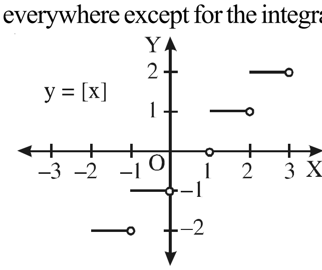
Example 4. Where is the function

$$f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1} \text{ continuous?}$$

Solution We know that the function y = ln x is continuous for x > 0 and $y = \tan^{-1}x$ is continuous on R. Thus, $y = \ln x + \tan^{-1} x$ is continuous on (0, ∞). The denominator, $y = x^2 - 1$, is a polynomial, so it is continuous everywhere. Therefore, f is continuous at all positive numbers x except where $x^2 - 1 = 0$. So f is continuous on the intervals (0, 1) and (1, ∞).

We need to examine the continuity of non-elementary functions carefully.

For instance, the greatest integer function f(x) = [x], is continuous everywhere except for the integral values of x.



y = [x] is discontinuous at $x \in I$.

Also, the fractional part function f(x) = {x}, is continuous everywhere except for the integral values of x.

Hence, the continuity of functions {f(x)} and [f(x)] should be checked at all points where f(x) becomes an integer.

The function y = sgn x (or $y = \frac{|x|}{x}$) is discontinuous at the point x = 0.

Hence, the continuity of the function sgn (f(x)) should be checked at the points where f(x) = 0. (Note that if f(x) is constanly equal to 0 when $x \to a$ then x = a may not be a point of discontinuity).

Usually there are only a few points in the domain of a given function f where a discontinuity can occur.

To find the points of discontinuity, we collect all the doubtful points and examine them for continuity.

We use the term **suspicious point** for a number a where

(i) The definition of the function changes or domain of f splits, or

(ii) substitution of x = a causes division by 0 in the function.

A function may be discontinuous at a suspicious point which can be found using the test of continuity.

For example, $y = \frac{x^2 - 1}{x^2 - 4}$ has two suspicious points $x = \pm 2$ (where the denominator becomes 0).

$y = x \sin\frac{1}{x}$ has one suspicious point x = 0.

For the function $y = |x^2 - 4|$ we have

$y = x^2 - 4$ when $x^2 - 4 \geq 0$

and $y = 4 - x^2$ when $x^2 - 4 < 0$.

This means the definition of the function changes when $x^2 - 4 = 0$, i.e. $x = \pm 2$. Thus, the function has two suspicious points $x = \pm 2$.

Note: There is no chance of continuity at points where the function is not defined. We know that the function cannot be continuous at such points.

Example 5. If $f(x) = [\sin \pi x], \quad 0 \leq x < 1$

$$= \left\{x - \frac{2}{3}\right\} \text{sgn}\left(x - \frac{5}{4}\right), 1 \leq x \leq 2,$$

where { . } represents the fractional function then find the suspicious points for continuity of function in the interval [0, 2].

Solution

(i) Continuity should be checked at the end-points of intervals of each definition i.e. x = 0, 1, 2.

(ii) For $[\sin \pi x]$, continuity should be checked at all values of x at which $\sin \pi x \in I$ i.e. x = 0, 1/2

(iii) For $\left\{x - \frac{2}{3}\right\} \text{sgn}\left(x - \frac{5}{4}\right)$, continuity should be checked when $x - 5/4 = 0$ i.e. $x = 5/4$ and when $x - 2/3 \in I$ i.e. x = 5/3.

Finally, continuity should be checked at the suspicious points x = 0, 1/2, 1, 5/4, 5/3 and 2 .

Example 6. Show that the function

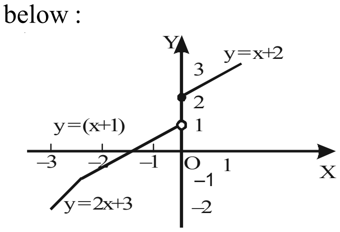
$$f(x) = \begin{cases} 2x + 3, & -3 \leq x < -2 \\ x + 1, & -2 \leq x < 0 \\ x + 2, & 0 \leq x \leq 1 \end{cases}$$

is discontinuous at x = 0 and continuous at other points in the interval [–3, 1].

Solution The graph of

$$f(x) = \begin{cases} 2x + 3, & -3 \leq x < -2 \\ x + 1, & -2 \leq x < 0 \\ x + 2, & 0 \leq x \leq 1 \end{cases}$$

is plotted below :



Here, we observe from the graph that at x = 0,

$\lim_{x \to 0^-} f(x) = 1$ and $\lim_{x \to 0^+} f(x) = 2$, which shows that the function is discontinuous at x = 0 and continuous at every other point in [–3, 1].

Example 7. Let

$$f(x) = \begin{cases} -2\sin x & \text{if} \quad x \leq -\pi/2 \\ A\sin x + B & \text{if} \quad -\frac{\pi}{2} < x < \pi/2 \\ \cos x & \text{if} \quad x \geq \pi/2 \end{cases}.$$

Find A and B so as to make the function continuous.

Solution At $x = -\pi/2$

$$\text{L.H.L.} = \lim_{x \to \frac{\pi}{2}^-} (-2\sin x) = 2$$

$$\text{R.H.L.} = \lim_{x \to -\frac{\pi}{2}^+} A\sin x + B = -A + B$$

So $B - A = 2$ ...(1)

At $x = \pi/2$

$$\text{L.H.L.} = \lim_{x \to \frac{\pi^-}{2}} A \sin x + B = A + B$$

$$\text{R.H.L.} = \lim_{x \to \frac{\pi^+}{2}} \cos x = 0$$

So $A + B = 0$ ...(2)

Solving (1) and (2) we get $A = -1$ and $B = 1$.

Example 8. Find the intervals on which f and g are continuous where $f(x) = \begin{cases} 3 - x & \text{if } -5 \le x < 2 \\ x - 2 & \text{if } 2 \le x < 5 \end{cases}$

and $g(x) = \begin{cases} 2 - x & \text{if } -5 \le x < 2 \\ x - 2 & \text{if } 2 \le x < 5 \end{cases}$.

Solution The domain for both function is [–5, 5). Both functions are continuous except possibly at the suspicious point x= 2. Examining f, we see

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - x) = 1$$

and $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 2) = 0$

so $\lim_{x \to 2} f(x)$ does not exist and f is discontinuous at x = 2.

Thus, f is continuous for $-5 \le x < 2$ and for $2 < x < 5$.

For g, we have g(2) = 0

and $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (2 - x) = 0$

and $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (x - 2) = 0$.

Therefore, $\lim_{x \to 2} g(x) = 0 = g(2)$, and g is continuous at x = 2. Hence, g is continuous throughout the interval [–5, 5).

Example 9. Find the points of discontinuity of the following functions for $x \in R$.

(i) $f(x) = \dfrac{1}{2\sin x - 1}$ (ii) $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

(iii) $f(x) = \dfrac{1}{x^4 + x^2 + 1}$ (iv) $f(x) = \dfrac{1}{1 - e^{\frac{x-1}{x-2}}}$

(v) $f(x) = [[x]] - [x - 2]$, where [.] represents the greatest integer function.

Solution A function is discontinuous at all such points where it is undefined.

(i) $f(x) = \dfrac{1}{2\sin x - 1}$

f(x) is discontinuous when $2 \sin x - 1 = 0$

i.e. $\sin x = \dfrac{1}{2} \Rightarrow x = 2n\pi + \dfrac{\pi}{6}, 2n\pi + \dfrac{5\pi}{6}, n \in I$

(ii) $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

f(x) is discontinuous when $x^2 - 3|x| + 2 = 0$

$\Rightarrow |x|^2 - 3|x| + 2 = 0$

$\Rightarrow (|x| - 1)(|x| - 2) = 0$

$\Rightarrow |x| = 1, 2$

$\Rightarrow x = \pm 1, \pm 2$

(iii) $f(x) = \dfrac{1}{x^4 + x^2 + 1} = \dfrac{1}{\left(x^2 + \frac{1}{2}\right)^2 + \frac{3}{4}}$

Now, $x^4 + x^2 + 1 = \left(x^2 + \dfrac{1}{2}\right)^2 + \dfrac{3}{4} \ge 1 \; \forall x \in R$

$\Rightarrow$ f(x) is continuous $\forall x \in R$

(iv) $f(x) = \dfrac{1}{1 - e^{\frac{x-1}{x-2}}}$

f(x) is discontinuous when $x - 2 = 0$ and also

when $1 - e^{\frac{x-1}{x-2}} = 0 \Rightarrow e^{\frac{x-1}{x-2}} = 1 \Rightarrow \dfrac{x-1}{x-2} = 0 \Rightarrow x = 1$.

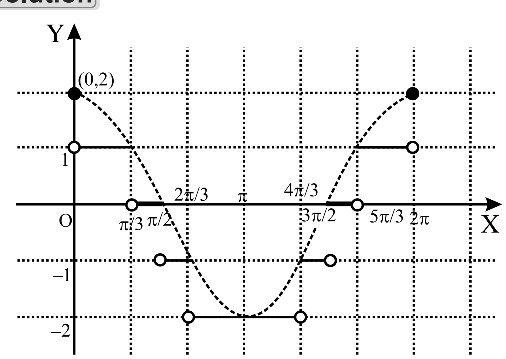
Thus f(x) is discontinuous at x = 1, 2.

(v) $f(x) = [[x]] - [x - 2] = [x] - ([x] - 2) = 2$

$\Rightarrow$ f(x) is continuous $\forall x \in R$

Example 10. Draw the graph and find the points of discontinuity for $f(x) = [2\cos x]$, $x \in [0, 2\pi]$, (where [.] represents the greatest integer function)

Solution



The suspicious points are:

when $2\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$;

when $2\cos x = \pm 1 \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$;

when $2\cos x = \pm 2 \Rightarrow x = 0, \pi, 2\pi$.

Clearly from the graph given above, f(x) is discontinuous at $x = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$.

Example 11. Draw the graph and discuss the continuity of $f(x) = [\sin x + \cos x], x \in [0, 2\pi]$, where [.] represents the greatest integer function.

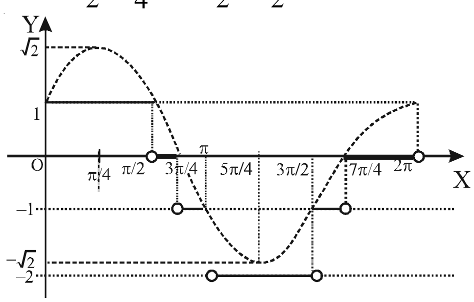
Solution $f(x) = [\sin x + \cos x]$

Let $g(x) = \sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

The range of g(x) is $[-\sqrt{2}, \sqrt{2}]$.

The suspicious points occur when $g(x) = 0, \pm 1$

$\Rightarrow \quad x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{2}, 2\pi$



Clearly from the graph given above f(x) is discontinuous at $x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{2}, 2\pi$.

Example 12. Discuss the continuity of f(x) in [0,2]

where $f(x) = \begin{cases} [\cos \pi x], & x \le 1 \\ |2x-3|[x-2], & x > 1 \end{cases}$

and [.] denotes the greatest integer function.

Solution For $x \in [0, 1]$, $f(x) = [\cos \pi x]$

Since [x] is discontinuous at $x \in I$, we must check continuity of $[\cos \pi x]$ at points where $\cos \pi x$ is an integer. This happens at $x = 0, \frac{1}{2}, 1$.

When $x \in [0, 1]$, the suspicious points are $x = 0, \frac{1}{2}, 1$.

Now for $x \in (1, 2)$, $\quad -1 < x - 2 < 0$

$\therefore \quad [x-2] = -1$ Thus, $f(x) = -|2x-3|$

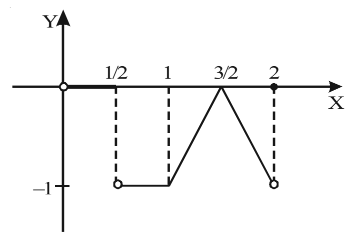
Since |x| is a continuous function, f(x) is continuous in (1, 2).

The right end point x = 2 is a suspicious point.

Hence, the suspicious points are $0, \frac{1}{2}, 1, \frac{3}{2}, 2$.

Now $\quad f(x) = \begin{cases} 1 & , \ x = 0 \\ 0 & , \ 0 < x \le \frac{1}{2} \quad (0 \le \cos \pi x < 1) \\ -1 & , \ \frac{1}{2} < x \le 1 \quad (-1 \le \cos \pi x < 0) \\ 2x-3 & , \ 1 < x \le \frac{3}{2} \quad (|2x-3| = 3-2x) \\ 3-2x & , \ \frac{3}{2} < x < 2 \quad (|2x-3| = 2x-3) \\ 0 & , \ x = 2 \quad ([x-2] = 0) \end{cases}$

Graph of f(x)



It is clear from the graph that f(x) is discontinuous at $x = 0, \frac{1}{2}, 2$ and continuous at all other points in [0, 2].

Example 13. If $f(x) = \begin{cases} \text{sgn}(x-2) \times [\ln x], & 1 \le x \le 3 \\ \{x^2\} & 3 < x \le 3.5 \end{cases}$

where [.] denotes the greatest integer function and {.} represents the fractional part function, find the point where the continuity of f(x) should be checked. Hence, find the points of discontinuity.

Solution Continuity should be checked at the endpoints of intervals of each definition, i.e., x = 1, 3, 3.5. $\{x^2\}$ is discontinuous for those values of x where x^2 is an integer. Hence, continuity should be checked when $x^2 = 10, 11, 12$ or $x = \sqrt{10}, \sqrt{11}, \sqrt{12}$.

sgn(x–2) should be checked when x –2 = 0 or x = 2.

[lnx] should be checked when lnx = 1 or x = e.

Hence, continuity must be checked at x =1, 2, e, 3, $\sqrt{10}, \sqrt{11}, \sqrt{12}$, 3.5.

Now, f(1) = 0 and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \text{sgn}(x-2) \times [\ln x] = 0$$

Hence f(x) is continuous at x = 1.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \text{sgn}(x-2)\times[\ln x] = (-1)\times 0 = 0$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \text{sgn}(x-2)\times[\ln x] = (1)\times 0 = 0$$

Hence, f(x) is continuous at x = 2.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \text{sgn}(x-2)\times[\ln x] = 1$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \{x^2\} = 0$$

Hence, f(x) is discontinuous at x = 3.

Also f(x) is discontinuous at

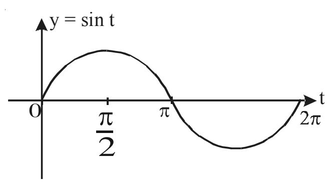
$x = \sqrt{10}, \sqrt{11}, \sqrt{12}$ because of $\{x^2\}$.

$$\lim_{x\to 3.5^-} f(x) = \lim_{x\to 3.5^-} \{x^2\} = 0.25 = f(3.5).$$

Hence, f(x) is discontinuous at $x = 3, \sqrt{10}, \sqrt{11}, \sqrt{12}$.

Example 14. Let $f(x) = \text{maximum}(\sin t, 0 \le t \le x), 0 \le x \le 2\pi$. Discuss the continuity of this function at $x = \frac{\pi}{2}$.

Solution $f(x) = \text{maximum}(\sin t, 0 \le t \le x), 0 \le x \le 2\pi$



If $x \in \left[0, \frac{\pi}{2}\right]$, sin t is an increasing function

Hence if $t \in [0, x]$, sin t will attain its maximum value at t = x.

$\therefore$ $f(x) = \sin x$ if $x \in \left[0, \frac{\pi}{2}\right]$

If $x \in \left(\frac{\pi}{2}, 2\pi\right]$ and $t \in [0, x]$

then sin t will attain its maximum value when $t = \frac{\pi}{2}$.

$\Rightarrow$ $f(x) = \sin\frac{\pi}{2} = 1$ if $x \in \left(\frac{\pi}{2}, 2\pi\right]$

$$\therefore \quad f(x) = \begin{cases} \sin x, & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ 1, & \text{if } x \in \left(\frac{\pi}{2}, 2\pi\right] \end{cases}$$

Now, $f\left(\frac{\pi}{2}\right) = 1$

$$\lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{x\to \frac{\pi}{2}^+} \sin x = 1$$

$$\lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{x\to \frac{\pi}{2}^+} 1 = 1.$$

Since $f(\frac{\pi}{2})$ = L.H.L. = R.H.L., f(x) is continuous at $x = \frac{\pi}{2}$.

Example 15. If the function

$$f(x) = \frac{2-(256-7x)^{1/8}}{(5x+32)^{1/5}-2},\ x \ne 0$$ is continuous everywhere then find the value of f(0).

Solution f(x) is continuous at all points except at the point where $(5x+32)^{1/5} = 2$ i.e. x = 0.

For continuity at x = 0,

$$f(0) = \lim_{x\to 0} f(x) = \lim_{h\to 0} f(0+h)$$

[note that here h assumes both positive and negative values close to 0]

$$= -\lim_{h\to 0} \frac{(256-7h)^{1/8}-(256)^{1/8}}{(5h+32)^{1/5}-(32)^{1/5}}$$

$$= -\lim_{h\to 0} \frac{\dfrac{(256-7h)^{1/8}-(256)^{1/8}}{(256-7h)-256}.(-7h)}{\dfrac{(5h+32)^{1/5}-(32)^{1/5}}{(5h+32)-32}.(5h)}$$

$$= \frac{7}{5}\lim_{h\to 0} \frac{\dfrac{(256-7h)^{1/8}-(256)^{1/8}}{(256-7h)-256}}{\dfrac{(5h+32)^{1/5}-(32)^{1/5}}{(5h+32)-32}}$$

$$= \frac{7}{5}.\frac{\frac{1}{8}.(256)^{1/8-1}}{\frac{1}{5}.(32)^{1/5-1}}$$

$$\left\{\because \lim_{x\to a}\frac{x^n-a^n}{x-a} - na^{n-1}\right\}$$

$$= \frac{7}{8}.\frac{(2)^{-7}}{(2)^{-4}} = \frac{7}{8}.\frac{1}{2^3} = \frac{7}{64}.$$

Hence, $f(0) = \frac{7}{64}$.

Example 16. If f(x) is a continuous function for all real values of x and satisfies $x^2 + (f(x)-2)x + 2\sqrt{3} - 3 - \sqrt{3}.f(x) = 0,\ \forall\ x \in R$, then find the value of $f(\sqrt{3})$.

Solution As f(x) is continuous for all x ∈ R.

$\lim_{x\to\sqrt{3}} f(x) = f(\sqrt{3})$ where

$$f(x) = \frac{x^2 - 2x + 2\sqrt{3} - 3}{\sqrt{3} - x}, x \neq \sqrt{3}$$

$$\lim_{x\to\sqrt{3}} f(x) = \lim_{x\to\sqrt{3}} \frac{x^2 - 2x + 2\sqrt{3} - 3}{\sqrt{3} - x}$$

$$= \lim_{x\to\sqrt{3}} \frac{(2-\sqrt{3}-x)(\sqrt{3}-x)}{(\sqrt{3}-x)} = 2(1-\sqrt{3})$$

∴ $f(\sqrt{3}) = 2(1 - \sqrt{3})$.

Example 17. Let y = f(x) be defined parametrically as $y = t^2 + t|t|$, $x = 2t - |t|$, t ∈ R. Then examine the continuity of f(x) at x = 0.

Solution $y = t^2 + t|t|$ and $x = 2t - |t|$

When t ≥ 0,

$x = 2t - t = t$, $y = t^2 + t^2 = 2t^2$

⇒ x = t and $y = 2t^2$

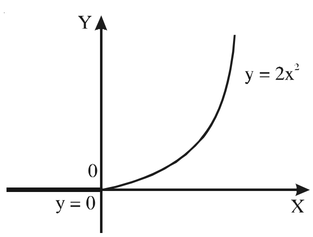
⇒ $y = 2x^2 \ \forall \ x \geq 0$

When t < 0,

⇒ $x = 2t + t = 3t$ and $y = t^2 - t^2 = 0$.

⇒ y = 0 for all x < 0.

Hence, $f(x) = \begin{cases} 2x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$ which is clearly continuous for all x as shown graphically below.



Example 18. Let $f(x) = x^3 - 3x^2 + 6 \ \forall \ x \in R$, and

$$g(x) = \begin{cases} \max f(t), x+1 \leq t \leq x+2, & -3 \leq x < 0 \\ 1-x, \quad \text{for} & x \geq 0 \end{cases}$$

Test continuity of g (x) for x ∈ [–3, 1].

Solution Since $f(x) = x^3 - 3x^2 + 6$

$f'(x) = 3x^2 - 6x = 3x(x-2)$

For maxima and minima, $f'(x) = 0$

⇒ x = 0, 2

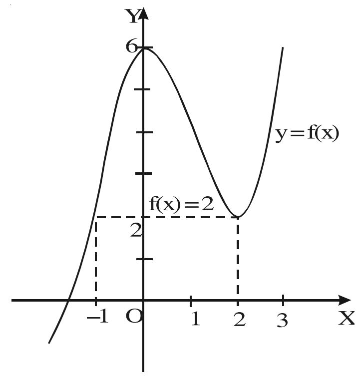
$f''(x) = 6x - 6$

$f''(0) = -6 < 0$ (local maxima at x = 0)

$f''(2) = 6 > 0$ (local minima at x = 2)

Also, f(0) = 6.

Now graph of f(x) is :



Clearly f(x) is increasing in (– ∞, 0) and (2, ∞) and decreasing in (0, 2).

Consider $x + 2 < 0 \Rightarrow x < -2$

For $-3 \leq x < -2$,

$-2 \leq x + 1 < -1$ and $-1 \leq x + 2 < 0$

Since f(x) increases, the maximum value of g(x) is f(x + 2).

∴ g(x) = f(x + 2) if $-3 \leq x < -2$.

Now consider x + 1 < 0 and $0 \leq x + 2 < 2$

For $-2 \leq x < -1$, g(x) = f(0).

Now for $x + 1 \geq 0$ and x + 2 < 2

i.e. $-1 \leq x < 0$, g(x) = f(x + 1).

$$\text{Hence, } g(x) = \begin{cases} f(x+2) & , \quad -3 \leq x < -2 \\ f(0) & , \quad -2 \leq x < -1 \\ f(x+1) & , \quad -1 \leq x < 0 \\ 1-x & , \quad x \geq 0 \end{cases}$$

Now, we can check that g(x) is continuous in the interval [–3, 1].

Example 19. Consider the function

$$f(x) = x\left[\frac{1}{x(1+x)} + \frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \ldots\infty\right] \text{ for } x \neq 0.$$

Find f(0) if f(x) is continuous at x = 0.

Solution We have the sum upto n terms as

$$\frac{1}{1+x} + \frac{(1+2x)-(1+x)}{(1+x)(1+2x)} + \frac{(1+3x)-(1+2x)}{(1+2x)(1+3x)}$$

$$+\ldots\ldots+\frac{(1+nx)-\left(1+\overline{n-1}\,x\right)}{\left(1+\overline{n-1}\,x\right)\ (1+nx)}$$

$$=\frac{2}{1+x}-\frac{1}{1+nx}\text{ upto } n \text{ terms when } x \neq 0.$$

Now, $f(x)=\frac{2}{1+x}$ if $x \neq 0$ and $n \to \infty$.

Thus, for continuity of f at x = 0, f(0) should be equal to the limiting value of f which is 2.

Example 20. Examine the continuity of the function

$$f(x)=\lim_{n\to\infty}\frac{\sin x}{1+(2\sin x)^{2n}} \text{ in } (0,\pi).$$

Solution $f(x)=\lim_{n\to\infty}\frac{\sin x}{1+(2\sin x)^{2n}}$

$$=\begin{cases}\sin x, & 0<x<\frac{\pi}{6} \text{ or } \frac{5\pi}{6}<x<\pi\\ \frac{1}{4}, & x=\frac{\pi}{6} \text{ or } \frac{5\pi}{6}\\ 0, & \frac{\pi}{6}<x<\frac{5\pi}{6}\end{cases}$$

We can see from the simplified definition of f that it is discontinuous at $x=\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Example 21. Given $f(x)=\sum_{r=1}^{n}(x^r+x^{-r})^2,\ x\neq\pm1$ and

$$g(x)=\begin{cases}\lim_{n\to\infty}\left(f(x)-2n)x^{-2n-2}(1-x^2)\right) & \text{for } x\neq\pm1\\ -1, & \text{for } x=\pm1\end{cases}$$

show that g(x) is continuous for all x in the domain.

Solution $f(x)=\sum_{r=1}^{n}\left(x^{2r}+\frac{1}{x^{2r}}+2\right)$

$$=\sum_{r=1}^{n}x^{2r}+\sum_{r=1}^{n}\frac{1}{x^{2r}}+2n$$

$$=(x^2+x^4+\ldots+x^{2n})+\left(\frac{1}{x^2}+\frac{1}{x^4}+\ldots\ldots+\frac{1}{x^{2n}}\right)+2n$$

$$=\frac{x^2(1-x^{2n})}{(1-x^2)}+\frac{1}{x^2}\frac{\left(1-\frac{1}{x^{2n}}\right)}{1-\frac{1}{x^2}}+2n$$

Thus, $f(x)=\frac{x^2(1-x^{2n})}{1-x^2}+\frac{\left(1-x^{2n}\right)}{(1-x^2)x^{2n}}+2n$

$$f(x)=\frac{(1-x^{2n})}{1-x^2}\left(x^2+\frac{1}{x^{2n}}\right)+2n,\ x\neq\pm1$$

$$\therefore\quad (f(x)-2n)(1-x^2)=(1-x^{2n})\left(x^2+\frac{1}{x^{2n}}\right)$$

Now consider

$$g(x)=\lim_{n\to\infty}\left(f(x)-2n)x^{-2n-2}(1-x^2)\right) \text{ for } x\neq\pm1$$

$$=\lim_{n\to\infty}(1-x^{2n})\left(x^2+\frac{1}{x^{2n}}\right)\cdot\frac{1}{x^{2n+2}},\ x\neq\pm1$$

$$=\lim_{n\to\infty}\frac{(1-x^{2n})(x^{2n+2}+1)}{(x^{2n})(x^{2n+2})}$$

$$=\lim_{n\to\infty}\left(-1-\frac{1}{x^{2n}}\right)\left(1+\frac{1}{x^{2n+2}}\right)$$

$$\text{Now } g(x)=\begin{cases}-1 & \text{if } |x|>1\\ \text{undefined} & \text{if } |x|<1\\ -1 & \text{if } x=\pm1\end{cases}$$

The domain of g(x) is $x\in(-\infty,-1]\cup[1,\infty)$.

Now clearly g is continuous for all x in the domain.

Example 22. Let $f: R \to R$ be a function which satisfies $f(x+y^3)=f(x)=(f(y))^3\ \forall\ x, y\in R$. If f is continuous at x = 0, prove that f is continuous everywhere.

Solution To prove $\lim_{h\to0} f(x+h)=f(x)$.

Put x = y = 0 in the given relation

$$f(0)=f(0)+(f(0))^3 \Rightarrow f(0)=0$$

Since f is continuous at x = 0, $\lim_{h\to0} f(h)=f(0)=0$.

Now, $\lim_{h\to0} f(x+h)=\lim_{h\to0} f(x)+(f(h))^3$

$$=f(x)+\lim_{h\to0}(f(h))^3 \qquad = f(x)+0=f(x).$$

Hence f is continuous for all $x\in R$.

Example 23. If $f(x+y)=f(x)\,.\,f(y)$ for all $x, y\in R$ and $f(x)=1+g(x)\,.\,G(x)$ where $\lim_{x\to0} g(x)=0$ and $\lim_{x\to0} G(x)$ exists, prove that f(x) is continuous at all $x\in R$.

Solution $\lim_{x\to0} g(x)=0 \quad\Rightarrow \lim_{h\to0} g(h)=0 \quad$ ...(1)

$\lim_{x\to0} G(x)$ exists $\Rightarrow \lim_{h\to0} G(0+h)=$ finite ...(2)

Now, $\lim_{h\to0} f(x+h)=\lim_{h\to0} f(x)\,.\,f(h)=f(x)\lim_{h\to0} f(h)$

$[\because f(x+y)=f(x)\,.\,f(y)]$

$$=f(x)\,.\lim_{h\to0}\{1+g(h)\,G(h)\},$$

[using the given relation]

$= f(x) . \{1 + \lim_{h\to 0} g(h) . \lim_{h\to 0} G(h)\}$
$= f(x) . \{1 + 0 . \text{finite}\}$, using (1) and (2)
$= f(x)$

$\therefore \lim_{h\to 0} f(x+h) = f(x).$

$\therefore$ f(x) is continuous everywhere.

Single Point Continuity

There are some functions which are continuous only at one point in an interval, though they are defined everywhere in the interval.

For example, $f(x) = \begin{cases} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{cases}$ is continuous only at x = 0. The limit of the function does not exist anywhere except at the point x = 0.

$f(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$ is continuous only at x = 0.

$f(x) = \begin{cases} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{cases}$ is continuous only at x = 1/2.

$f(x) = \begin{cases} x^2 & \text{if } x \in Q \\ 1 & \text{if } x \notin Q \end{cases}$ is continuous only at x = 1 or – 1.

Example 24. Find the values of a and b if the function $f(x) = x^2 + ax + 1$, if x is rational
$= ax^2 + bx + 1$, if x is irrational
is continuous at x = 1 and 2.

Solution Continuity at x = 1 implies
$a + 2 = a + b + 1$
which gives $b = 1$.
Continuity at x = 2 implies
$2a + 5 = 4a + 2b + 1$
which gives $a = 1$.

Note: Point functions are treated as discontinuous. If the domain of the function contains a countable number of points then it is discontinuous at all of these points.
For example, $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1. Here the function is defined only at x = 1.

Also, $f(x) = \dfrac{1}{\{x\} + \{-x\} - 1}$ is defined only for integers.
Hence, it is discontinuous at all points in the domain.

Concept Problems B

1. At what points are the tangent and cotangent functions continuous?
2. Where are the following functions discontinuous?
   (i) sec x (ii) $\dfrac{1+x^2}{1-x^2}$
3. Prove that the function
   $f(x) = \begin{cases} x+1, & -1 \le x \le 0, \\ -x, & 0 < x \le 1 \end{cases}$ is discontinuous.
4. For what value of 'k' is the function,
   $f(x) = \begin{cases} \dfrac{\sin 5x}{3x}, & x \ne 0 \\ k, & x = 0 \end{cases}$ continuous ?
5. Is the function
   $f(x) = \begin{cases} 2x+1 & \text{if } -3 < x < -2 \\ x-1 & \text{if } -2 \le x < 0 \\ x+2 & \text{if } 0 \le x < 1 \end{cases}$
   continuous everywhere in (–3, 1) ?
6. If b and c are given, find all values of a for which
   $f(x) = \begin{cases} 2\cos x & \text{if } x \le c, \\ ax^2 + b & \text{if } x > c \end{cases}$ is continuous.
7. Find constants a and b so that the given function will be continuous for all x in the domain.
   (i) $f(x) = \begin{cases} \dfrac{\tan ax}{bx} & \text{if } x < 0 \\ 4 & \text{if } x = 0 \\ ax + b & \text{if } x > 0 \end{cases}$
   (ii) $f(x) = \begin{cases} 2\sin(a\cos^{-1} x) & \text{if } -1 \le x < 0 \\ \sqrt{3} & \text{if } x = 0 \\ ax + b & \text{if } x > 0 \end{cases}$
8. For what (if any) values of x are the following functions discontinous:
   (i) $y = [x^2]$, (ii) $y = [\sqrt{x}]$,
   (iii) $y = \sqrt{(x - [x])}$, (iv) $y = [2x]$,
   (v) $y = [x] + [-x]$?

9. (i) For $x \neq 0$, let $f(x) = [1/x]$, where [.] denotes the greatest integer function. Sketch the graph of f over the intervals $[-2, -\frac{1}{5}]$ and $[\frac{1}{5}, 2]$. What happens to f(x) as $x \to 0$ through positive values? and through negative values? Can you define f(0) so that f becomes continuous at 0?

(ii) Answer the same when $f(x) = (-1)^{[1/x]}$ for $x \neq 0$.

(iii) Answer the same when $f(x) = x(-1)^{[1/x]}$ for $x \neq 0$.

10. If f(x) is continuous on $(-\infty, \infty)$, then it has no vertical asymptotes. True or false.

11. Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither ?

(i) $f(x) = \begin{cases} 1+x^2 & \text{if } x \le 0 \\ 2-x & \text{if } 0 < x \le 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$

(ii) $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x > 1 \end{cases}$

12. Choose parameters, entering into the definition of the function such that the function f(x) becomes continuous:

(i) $f(x) = \begin{cases} 2^{1/(x-1)}, & x < 1 \\ ax^2+bx+1, & x \ge 1 \end{cases}$

(ii) $f(x) = \begin{cases} x^2+x+1, & x \ge -1 \\ \sin \pi(x+a)), & x < -1. \end{cases}$

13. Locate the discontinuities of the following functions:

(i) $y = \dfrac{1}{1+e^{1/x}}$ (ii) $y = \ln(\tan^2 x)$

14. Let $f(x) = 2^{-1/x^2}$ for $x \neq 0$.

(i) Find $\lim_{x\to\infty} f(x)$ (ii) Find $\lim_{x\to 0} f(x)$

(iii) Is it possible to define f(0) in such a way that f is continuous for all real x ?

15. Determine the set of all points where the function $f(x) = \dfrac{x^3}{4+|x|}$ is continuous.

16. Examine the continuity of the function over $x \in R$.

(i) $f(x) = \tan \dfrac{1}{x}$ (ii) $f(x) = \dfrac{x-1}{x(x+1)(x^2-4)}$

(iii) $f(x) = [x] + [-x]$.

Practice Problems B

17. Let $f(x) = \begin{cases} \dfrac{1-\cos(4x)}{x^2}, & x < 0 \\ a, & x = 0 \\ \dfrac{\sqrt{x}}{(16+\sqrt{x})^{1/2}-4}, & x > 0 \end{cases}$

then find the value of 'a' for which f(x) is continuous at x = 0.

18. If $f(x) = \text{sgn}\left(\left\{x - \frac{1}{2}\right\}\right)[\ln x]$, $1 < x \le 3$

$= \{x^2\}$, $3 < x \le 3.5$

Find the point where the continuity of f(x) should be checked.

19. The function f(x) is defined as

$f(x) = \begin{cases} x^2 \cos e^{1/x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$.

Show that f(x) is discontinuous at x = 0.

20. If

$f(x) = \begin{cases} x & \text{for rational values of } x \text{ in } [0, 1] \\ 1-2x & \text{for irrational values of } x \text{ in } [0, 1] \end{cases}$

Show that f(x) is continuous only at x = 1/3.

21. Find the points of continuity of $f(x) = \sqrt{\dfrac{x^2-2x}{x-1}}$.

22. If the function f given by

$f(x) = \begin{cases} \dfrac{\tan 2ax}{3bx} & \text{if } x < 0 \\ c & \text{if } x = 0 \\ \dfrac{|x|}{x} + \dfrac{x^3-1}{x-1} & \text{if } 0 < x < 1 \\ b & \text{if } x \ge 1 \end{cases}$

is everywhere continuous, find a, b, and c.

23. Let $f:[0,\infty)\to[0,\infty)$,

$f(x)=\sqrt{x+\sqrt{x+\sqrt{x+.....}}}$. Is f right continuous at 0?

24. What are the points of discontinuity of the function

$$f(x)=\begin{cases}\cos x & \text{if } x\in Q\\ \sin x & \text{if } x\notin Q\end{cases}?$$

25. Let f be the "nearest integer, with rounding down" function. That is,

$$f(x)=\begin{cases}\text{the integer nearest to } x \text{ if } x \text{ is not midway between two consecutive integers}\\ x-\frac{1}{2} \text{ if } x \text{ is midway between two consecutive integers}\end{cases}$$

(i) Does $\lim_{x\to 3.5^-} f(x)$ exist ? If so, evaluate it.

(ii) Does $\lim_{x\to 3.5} f(x)$ exist ? If so, evaluate it.

(iii) Is f continuous at 3.5 ?

(iv) Where is f not continuous ?

26. Test the continuity of the function

$$y=\lim_{n\to\infty}\frac{x}{1+(2\cos x)^{2n}}.$$

27. If f(x) = x + {– x} + [x] ,where [.] denotes the greatest integer function. and {.} denotes the fractional part function. Discuss the continuity of f in [–2 , 2] .

28. Discuss the continuity of the function $f(x)=[[x]]-[x-1]$, where [.] denotes the greatest integer function.

29. Find all possible values of a and b so that f (x) is continuous for all $x\in R$ if

$$f(x)=\begin{cases}|ax+3|, & \text{if } x\le -1\\ |3x+a|, & \text{if } -1<x\le 0\\ \frac{b\sin 2x}{x}-2b, & \text{if } 0<x<\pi\\ \cos^2 x-3, & \text{if } x\ge\pi\end{cases}.$$

30. Examine the continuity at x= 0 of the function

$$f(x)=\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+....\infty.$$

2.3 CLASSIFICATION OF DISCONTINUITY

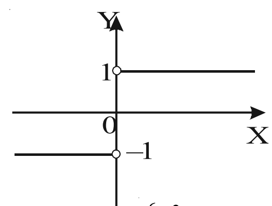
Discontinuity of first and second kind

Let a function f be defined in the neighbourhood of a point a, except perhaps at a itself. Also let both the one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, but the conditions of continuity are not satisfied. Then the function f(x) is said to have a **discontinuity of the first kind** at the point a.

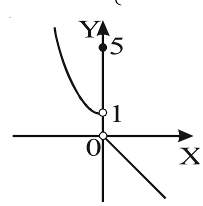
For instance, the function $f(x)=\begin{cases}\dfrac{x}{1+2^{\frac{1}{x}}}, & x\neq 0\\ 1, & x=0\end{cases}$

has a discontinuity of the first kind at x = 0, since both the one-sided limits are 0 but this is not equal to f(0) which is 1.

The function $f(x)=\frac{x}{|x|}$ has a discontinuity of the first kind at x = 0 since the left hand limit is $\lim_{x\to 0^-} f(x)=-1$, while the right hand limit is $\lim_{x\to 0^+} f(x)=1$ i.e. the one sided limits exists.



The function $y=\begin{cases}x^2+1 & \text{for } x<0,\\ 5 & \text{for } x=0,\\ -x & \text{for } x>0,\end{cases}$



has a discontinuity of the first kind at x = 0 because the one-sided limits exist.

A function f(x) having a finite number of discontinuities of first kind in a given interval is called sectionally or **piecewise continuous** function.

Sometimes, a function f is discontinuous at x = a because $\lim_{x\to a} f(x)$ does not exist as one or both of the one-sided limits do not exist.

The function f(x) is said to have **discontinuity of the second kind** at x = a, if atleast one of the one-sided limits (L.H.L. or R.H.L.) at the point x = a does not exist or equals to infinity.

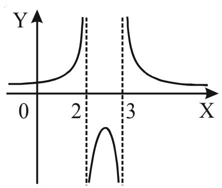
Let us show that for the function $f(x)=\dfrac{1}{3-x}$, the point x = 3 is a point of discontinuity of the second kind.

Consider the limits on the left and on the right at the point x = 3:

$$\lim_{x\to 3^-}\frac{1}{3-x}=\infty,\ \lim_{x\to 3^+}\frac{1}{3-x}=-\infty$$

Thus, the one-sided limits of the function f(x) at the point x = 3 are infinite, and according to the definition, this means that the function f(x) has a point of discontinuity of the second kind at this point.

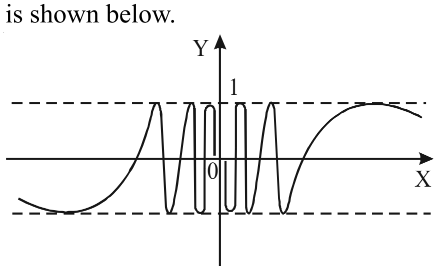
The function $y=\dfrac{1}{(x-2)(x-3)}$ has no one-sided limits at x = 2 and x = 3. Therefore x = 2 and x = 3 are discontinuities of the second kind. We can see the graph of the function below.



The function y = ln |x| at the point x = 0 has the limit $\lim_{x\to 0} \ln|x|=-\infty$. Consequently, $\lim_{x\to 0} f(x)$ (and also the one-sided limits) does not exist. Hence, x = 0 is a discontinuity of the second kind.

It is not true that discontinuities of the second kind only arise when $\lim_{x\to a} f(x)=\infty$. The situation can be different. The function y = sin (1/x), does not have the one-sided limits as $x \to 0$ since the values of the function sin (1/x) do not approach a certain number, but oscillate an infinite number of times within the interval from –1 to 1 as $x\to 0$.

It has a discontinuity of the second kind at x = 0. The graph is shown below.



Removable Discontinuity

A function f is said to have a removable discontinuity at x = a, if $\lim_{x\to a} f(x)$ exists but is not equal to f(a). In this case we can redefine the function such that $\lim_{x\to a} f(x)=f(a)$ and make it continuous at x = a.

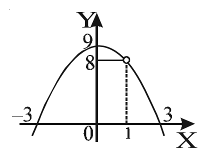
For example, f(x) = [x] + [–x] has a removable discontinuity at x = 1 since $\lim_{x\to 1} f(x)=-1$, but it is not equal to f(1) which is 0.

Removable discontinuity can be further classified as:

(i) Missing Point Discontinuity

A function f is said to have a missing point discontinuity at x = a if $\lim_{x\to a} f(x)$ exists and the function is undefined at x = a.

For example, $f(x)=\dfrac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at x = 1 since $\lim_{x\to 1} f(x)=8$ but the function is undefined at x = 1.



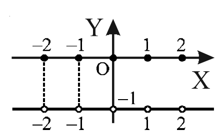
(ii) Isolated Point Discontinuity

A function f is said to have an isolated point discontinuity at x = a if $\lim_{x\to a} f(x)$ exists and the function is defined at x = a, but they are unequal.

For example, $f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & x\in R-\{2\}\\ 1, & x=2\end{cases}$ has an isolated point discontinuity at x = 2 since $\lim_{x\to 2} f(x)$ exists and is equal to 4, but it does not match with the function's value at x = 2, which is 1.

Also the function $f(x)=[x]+[-x]=\begin{cases}0 & \text{if } x\in I\\ -1 & \text{if } x\notin I\end{cases}$

has isolated point discontinuities at all integral x as shown in the figure

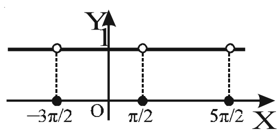


The function f(x) = sgn(cos2x–2sinx+3)

$$= \text{sgn}(\,2(2+\sin x)(1-\sin x)\,)$$

$$= \begin{cases} 0 & \text{if } x = 2n\pi + \dfrac{\pi}{2} \\ 1 & \text{if } x \neq 2n\pi + \dfrac{\pi}{2} \end{cases}$$

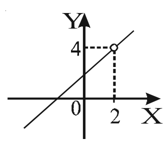
has an isolated point discontinuity as $x = 2n\pi + \dfrac{\pi}{2}$.



Continuous extension of a function

A function F is called a continuous extension of a given function f, at a removable discontinuity a of f, if the values of F and f agree at every point of the domain of f and the value of F at a is the limit of f at a.

For example, let $f(x) = (x^4 - 4)/(x-2),\ x \in R - \{2\}$. Then x = 2 is a missing point discontinuity.



But since $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2}(x+2) = 4$,

we see that x = 2 is a removable discontinuity.

Here F is defined as

$$F(x) = \begin{cases} \dfrac{x^2-4}{x-2} & ,\ x \in R - \{2\} \\ 4 & ,\ x = 2 \end{cases}$$

It is the continuous extension of f at x = 2.

The function $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & ,\ x \in R - \{3\} \\ 4 & ,\ x = 2 \end{cases}$

is defined for all $x \in R$. At x = 3 it has the following left hand and right hand limits:

$$\lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3-0}(x+3) = 6,$$

$$\lim_{x\to 3^+} \frac{x^2-9}{x-3} = \lim_{x\to 3-0}(x+3) = 6,$$

that is, the one-sided limits are equal. Hence, $\lim\limits_{x\to 3} f(x) = 6$.

However, the point x=3 is a discontinuity in the function since the limit is not equal to the function's value.

Let us see how we can make the function continuous at x = 3. Consider the function F(x) defined as

$$F(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$$

The values of F(x) coincide with those of the function $f(x) = (x^2-9)/(x-3)$ everywhere except at x = 3, where the "old" function was defined as 4, while at x = 3 the "new" function has a value of 6.

The function F(x) fulfills the equality

$$\lim_{x\to 3} F(x) = F(3) = 6,$$

which means that F(x) is continuous at x = 3. Thus, the discontinuity of f(x) has been removed.

Note that we were able to get a continuous function F(x) from the discontinuous function f(x) because the one-sided limits were equal.

Irremovable Discontinuity

A function f is said to have an irremovable discontinuity at x = a, if $\lim\limits_{x\to a} f(x)$ does not exist. In this case we cannot redefine the function such that $\lim\limits_{x\to a} f(x) = f(a)$ and make it continuous at x = a.

Irremovable discontinuities can be further classified as :

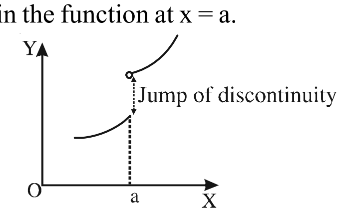
(i) Finite discontinuity

A function f is said to have a finite or jump discontinuity at x = a if $\lim\limits_{x\to a} f(x)$ does not exist since the left hand limit and the right hand limit are unequal, but the one-sided limits do exist.

For instance, f(x) = [x] has a finite or jump discontinuity at all integral x.

Jump of a function at a point

If x = a is a point of finite discontinuity of the function f(x), then the graph of this function undergoes a jump at x = a. The difference R.H.L. – L.H.L. i.e. $f(a^+) - f(a^-)$ is called the jump in the function at x = a.



The graph of the function f(x) = sgn(x) makes a jump of 2 units at the point x = 0 since

$f(0^+) - f(0^-) = 1 - (-1) = 2.$

The difference between the greatest and least of the three numbers $f(a^+)$, $f(a^-)$, f(a) is the **saltus** or measure of discontinuity of the function at the point a.

(i) $\lim_{x\to 0} \tan^{-1}\left(\frac{1}{x}\right)$ $\begin{cases} f(0^+) = \frac{\pi}{2} \\ f(0^-) = -\frac{\pi}{2} \end{cases}$; jump = π

(ii) $\lim_{x\to\pi} \frac{|\sin x|}{x-\pi}$ $\begin{cases} f(\pi^+) = -1 \\ f(\pi^-) = 0 \end{cases}$; jump = –1

(iii) $\lim_{x\to 2} \frac{[x]}{x}$ $\begin{cases} f(2^+) = 1 \\ f(2^-) = \frac{1}{2} \end{cases}$; jump = $\frac{1}{2}$

(iv) $f(x) = \frac{1}{1-e^{-1/x}}$ and f(0) = e .

$$f(0^+) = \lim_{h\to 0}\frac{1}{1-e^{-1/(0+h)}} = \lim_{h\to 0}\frac{1}{1-e^{-1/h}}$$

$$= \frac{1}{1-e^{-\infty}} = \frac{1}{1-0} = 1$$

and $$f(0^-) = \lim_{h\to 0}\frac{1}{1-e^{-1/(h-h)}} = \lim_{h\to 0}\frac{1}{1-e^{1/h}}$$

$$= \frac{1}{1-e^{\infty}} = \frac{1}{1-\infty} = 0 .$$

Hence f is discontinuous at x = 0, the discontinuity being of the first kind and irremovable. This function has a jump of one unit at 0 since $f(0^+) - f(0^-) = 1$. The saltus is e.

(ii) Infinite discontinuity

A function is said to have an infinite discontinuity at x = a if atleast one of the one-sided limits is infinite. For instance, the following functions have infinite discontinuity:

(i) $f(x) = \frac{x}{1-x}$ at x = 1 $\begin{cases} f(1^+) = -\infty \\ f(1^-) = \infty \end{cases}$

(ii) $f(x) = 2^{\tan x}$ at $x = \frac{\pi}{2}$ $\begin{cases} f\left(\frac{\pi}{2}^+\right) = 0 \\ f\left(\frac{\pi}{2}^-\right) = \infty \end{cases}$

(iii) $f(x) = \frac{1}{x^2}$ at x = 0 $\begin{cases} f(0^+) = \infty \\ f(0^-) = \infty \end{cases}$

(iv) $f(x) = \begin{cases} x^2 \ , \ x \le 0 \\ \ln x, \ x > 0 \end{cases}$

has an infinite discontinuity at x = 0 since the right hand limit is infinite (note that the left hand limit is 0.

Pole discontinuities

The concept of pole discontinuity is related with infinite limit. For a point x = a to qualify as a pole of a function f, we must have $\lim_{x\to a}\frac{1}{f(x)} = 0$.

What this means roughly is that "f(x) becomes numerically big and stays big as x gets close to a".

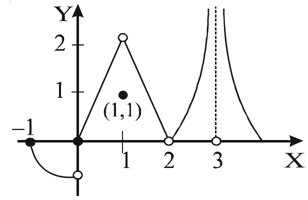
The function $y = \frac{1}{(x-2)(x-3)}$ has pole at x = 2 and x = 3. Note that $f(x) = \begin{cases} x^2 \ , \ x \le 0 \\ \ln x, \ x > 0 \end{cases}$ does not have a pole discontinuity at x = 0, even if it has an infinite discontinuity at x = 0 since the left hand limit is finite.

Again, the reciprocal of a polynomial of degree n has atmost n poles : in fact, the real zeros of a polynomial are the poles of its reciprocal.

In particular, the reciprocal of the quadratic polynomial with negative discriminant has no poles. The poles of a rational function in "lowest terms" are the zeros of the denominator.

The zeros of sin x and cos x are respectively the poles of cosec x and sec x.

Consider the following graph to understand the nature of discontinuity.



From the above graph note that

(i) f is continuous at x = – 1
(ii) f has non removable (finite type) discontinuity at x = 0
(iii) f has isolated discontinuity at x = 1
(iv) f has missing point discontinuity at x = 2
(v) at x = 3 non removable infinite type discontinuity.

(iii) Oscillatory discontinuity

A function is said to have an oscillatory discontinuity at x = a if atleast one of the one-sided limits does not exist becase of too much oscillation in the values of the function.

For example, $f(x) = \sin\frac{1}{x}$ has an oscillatory discontinuity at $x = 0$.

The function $f(x) = \left[1+\frac{1}{3}\sin(\ln|x|)\right]$ oscillates between 0 and 1 at x = 0 and hence has a oscillatory discontinuity at x = 0.

Note: In all the cases of irremovable discontinuity the value of the function at x = a (point of discontinuity) i.e. f(a) may or maynot be defined.

Example 1. If $f(x)=\begin{cases} x, & x<1 \\ x^2, & x>1 \end{cases}$

then find the type of discontinuity at x = 1.

Solution $f(x)=\begin{cases} x, & x<1 \\ x^2, & x>1 \end{cases}$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x = 1$$

and $$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} x^2 = 1$$

$$\therefore \quad \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = 1$$

but f(1) is not defined.

So, f(x) has a missing point removable discontinuity at x = 1. The discontinuity is of first kind.

Example 2. Let $f(x) = \cos^{-1}\{\cot x\},\ x<\frac{\pi}{2}$

$$= \pi[x]-1, \quad x \ge \frac{\pi}{2}$$

Find jump of discontinuity at $x = \frac{\pi}{2}$.

Solution $f(x)=\begin{cases} \cos^{-1}\{\cot x\} & \text{if } x<\frac{\pi}{2} \\ \pi[x]-1 & \text{if } x\ge\frac{\pi}{2} \end{cases}$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-} \cos^{-1}\{\cot x\}$$

$$= \cos^{-1} 0 = \frac{\pi}{2}.$$

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+} \pi[x]-1 = \pi-1.$$

∴ The jump of discontinuity = R.H.L. – L.H.L.

$$= \pi - 1 - \frac{\pi}{2} = \frac{\pi}{2} - 1.$$

Example 3. Let $f(x) = \frac{2\cos x - \sin 2x}{(\pi-2x)^2}$,

$g(x) = \frac{e^{-\cos x}-1}{8x-4\pi}$, and $h(x) = \begin{cases} f(x), & x<\pi/2 \\ g(x), & x>\pi/2 \end{cases}$

then show that h has an irremovable discontinuity at x = π/2.

Solution $$h(x) = \begin{cases} \dfrac{2\cos x - \sin 2x}{(\pi-2x)^2}, & x<\dfrac{\pi}{2} \\ \dfrac{e^{-\cos x}-1}{8x-4\pi}, & x>\dfrac{\pi}{2} \end{cases}$$

L.H.L. at x = π/2

$$= \lim_{h\to0} \frac{2\sin h - \sin 2h}{4h^2}$$

$$= \lim_{h\to0} \frac{2\sin h(1-\cos h)}{4h^2} = 0.$$

$$\text{R.H.L.} = \lim_{h\to0} \frac{e^{\sin h}-1}{\theta((\pi/2)+h)-4\pi}$$

$$= \lim_{h\to0} \frac{e^{\sin h}-1}{8h}\cdot\frac{\sin h}{\sin h} = \frac{1}{8}.$$

Thus, $h\left(\frac{\pi}{2}^+\right) = 0$ and $h\left(\frac{\pi}{2}^-\right) = \frac{1}{8}$

We have $h\left(\frac{\pi}{2}^+\right) \ne h\left(\frac{\pi}{2}^-\right)$.

Since L.H.L. ≠ R.H.L., h(x) has an irremovable discontinuity at x = π/2.

Example 4. State the number of point of discontinuity and discuss the nature of discontinuity for the function $f(x) = \frac{1}{\ln|x|}$ and also sketch its graph.

Solution $f(x)=\begin{cases}\dfrac{1}{\ln x} & \text{if } x>0,\ x\neq 1\\ \dfrac{1}{\ln(-x)} & \text{if } x<0,\ x\neq -1\end{cases}$

The function is obviously discontinuous at x = 0, 1, –1. as it is not defined.

$$\left.\begin{matrix}\lim_{x\to 0^+} f(x)=0\\ \lim_{x\to 0^-} f(x)=0\end{matrix}\right\}\text{ Limit exists at } x=0.$$

Hence there is a removable discontinuity (missing point) at x = 0.

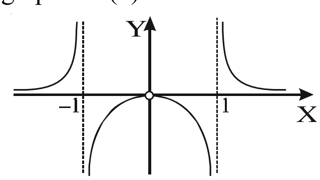
$$\left.\begin{matrix}\lim_{x\to 1^+} f(x)=\infty\\ \lim_{x\to 1^-} f(x)=-\infty\end{matrix}\right\}\text{ Limit dne.}$$

Hence there is a nonremovable discontinuity (infinite type) at x = 1.

$$\left.\begin{matrix}\lim_{x\to -1^+} f(x)=-\infty\\ \lim_{x\to -1^-} f(x)=\infty\end{matrix}\right\}\text{ Limit dne.}$$

Hence there is a non removable discontinuity (infinite type) at x = –1.

Note that f (x) is even $\Rightarrow$ the graph is symmetric about y axis. The graph of f (x) is as follows.



Example 5. Let $f(x)=\lim_{n\to\infty}\dfrac{1}{1+n\sin^2 x}$, then find $f\left(\dfrac{\pi}{4}\right)$ and also comment on the continuity at x = 0.

Solution Let $f(x)=\lim_{n\to\infty}\dfrac{1}{1+n\sin^2 x}$

$$f\left(\frac{\pi}{4}\right)=\lim_{n\to\infty}\frac{1}{1+n\,.\,\sin^2\frac{\pi}{4}}$$

$$=\lim_{n\to\infty}\frac{1}{1+n\left(\frac{1}{2}\right)}=0$$

Now,

$$f(0)=\lim_{n\to\infty}\frac{1}{n\,.\,\sin^2(0)+1}=\frac{1}{1+0}=1$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\left[\lim_{n\to\infty}\frac{1}{1+n\sin^2 x}\right]\{\text{form}\left[\frac{1}{1+\infty}\right]\}=0$$

Here $\sin^2 x$ is a very small quantity but not zero.

$\therefore$ Thus f(x) is discontinuous at x = 0. The type of discontinuity is isolated point removable discontinuity.

Example 6. Examine the function $f(x)=\lim_{n\to\infty}\dfrac{x}{1+(4\sin^2 x)^n}$ for continuity in $(0,\pi)$. Plot its graph and state the nature of discontinuity.

Solution For $x=\dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$, $4\sin^2 x=1$

$$\therefore\quad f(x)=\lim_{n\to\infty}\frac{x}{1+1^n}=\frac{x}{2}.$$

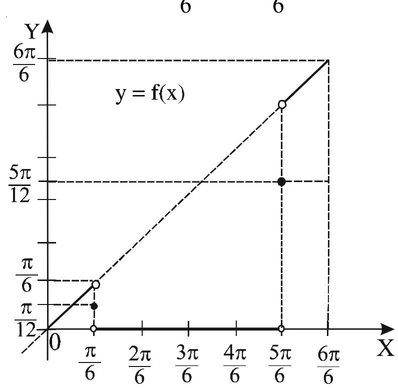
For $\dfrac{\pi}{6}<x<\dfrac{5\pi}{6}$, $4\sin^2 x>1$

$$\therefore\quad f(x)=\lim_{n\to\infty}\frac{x}{1+(\text{greater than }1)^n}=0.$$

For $0\le x<\dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}<x\le\pi$, $4\sin^2 x<1$

$$\therefore\quad f(x)=\lim_{n\to\infty}\frac{x}{1+(\text{less than }1)^n}=x.$$

$$\text{Hence } f(x)=\begin{cases}x & \text{for } 0\le x<\frac{\pi}{6} \text{ or } \frac{5\pi}{6}<x\le\pi\\ \frac{x}{2} & \text{for } x=\frac{\pi}{6} \text{ or } \frac{5\pi}{6}\\ 0 & \text{for } \frac{\pi}{6}<x<\frac{5\pi}{6}\end{cases}$$



The graph of f(x) is as shown above.

From the graph it is clear that f (x) is continuous everywhere in $(0,\pi)$ except at $\dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$ and where it has removable discontinuity of finite type.

Example 7. Consider f(x)

$$= \frac{1}{\sqrt{b-a}} \frac{\sqrt{\frac{b-a}{a}}\sin 2x}{\sqrt{1+\left(\sqrt{\frac{b-a}{a}}\sin x\right)^2}}\sqrt{a+b\tan^2 x},$$

for $b > a > 0$ and $g(x) = \text{sgn}(f(x))$. Find whether g (x) is continuous at x = 0 or not and state the nature of discontinuity, if discontinuous.

Solution $$f(x) = \frac{1}{\sqrt{b-a}}\cdot\frac{\sqrt{b-a}\cdot\sin 2x\cdot\sqrt{a+b\tan^2 x}}{\sqrt{a}\cdot\sqrt{\frac{a+(b-a)\sin^2 x}{a}}}$$

$$= \frac{\sin 2x\cdot\sqrt{a+b\tan^2 x}}{\sqrt{a\cos^2 x + b\sin^2 x}}$$

$$= \frac{\sin 2x\cdot\sqrt{a\cos^2 x+b\sin^2 x}}{|\cos x|\sqrt{a\cos^2 x+b\sin^2 x}}$$

Hence, $f(x) = \frac{\sin 2x}{|\cos x|}$

Now, $g(x) = \text{sgn}\left(\frac{\sin 2x}{|\cos x|}\right)$

We have $g(0) = 0$

$$g(0^+) = \lim_{h\to 0}\left(\frac{\sin 2h}{\cos h}\right) = \lim_{h\to 0}\text{sgn}(2\sin h) = 1.$$

$$g(0^-) = \lim_{h\to 0}\text{sgn}\left(\frac{-2\sin 2h}{\cos h}\right) = \lim_{h\to 0}\text{sgn}(-2\sin h)$$

$= -1$

Hence, $g(x) = \text{sgn}(f(x))$ is discontinuous at x = 0. The nature of discontinuity is irremovable discontinuity of finite type.

Example 8. Let

$$f(x) = \begin{cases} \dfrac{e^x - 1}{\sqrt{(1+x^2)-\sqrt{1-x^2}}} & \text{if } x \neq 0 \\ \sqrt{\dfrac{2}{3}} & \text{if } x = 0 \end{cases},$$

then find whether f(x) is continuous at x = 0 or not and state the nature of discontinuity, if discontinuous.

Solution

$$\text{Limit} = \lim_{x\to 0}\frac{\left(\frac{e^x-1}{x}\right)\cdot x\cdot\sqrt{(1+x^2)+\sqrt{1-x^2}}}{\sqrt{(1+x^2)-\sqrt{1-x^2}}\cdot\sqrt{(1+x^2)+\sqrt{1-x^2}}}$$

$$= \lim_{x\to 0}\frac{\sqrt{2}\cdot x}{\sqrt{(1+x^2)^2-(1-x^2)}} = \lim_{x\to 0}\frac{\sqrt{2}\,x}{\sqrt{x^4+3x^2}}$$

Now R.H.L. $= \frac{\sqrt{2}}{\sqrt{3}}$ and L.H.L. $= -\frac{\sqrt{2}}{\sqrt{3}}$

Hence, f(x) is discontinuous at x = 0. The nature of discontinuity is irremovable discontinuity of finite type.

Concept Problems C

1. Let f(x) $= x$, $x < 1$
$= x^2$, $x > 1$
$= 2$, $x = 1$
Find the type of discontinuity at x = 1.

2. Let f(x) $= x$, $x < 1$
$= 2x$, $x \geq 1$
Find the type of discontinuity at x = 1.

3. Prove that the function $h(x) = 2^{\tan x}$ has infinite discontinuity at $x = \pi/2$.

4. If $f(x) = x + \frac{x+2}{|x+2|}$ find the points of discontinuity and determine the jumps of the function at these points.

5. If $f(x) = \frac{1}{x-a}\sin\frac{1}{x-a}$ find $f(a^+)$ and $f(a^-)$. Is the function continuous at x = a?

6. Test the discontinuity of the following function at x = a and specify the type of discontinuity.

$$f(x) = \frac{1}{x-a}\text{cosec}\frac{1}{x-a},\ x \neq a \text{ and } f(a) = 0.$$

7. The function $f(x) = \tan^{-1}\frac{1}{x-2}$ is meaningless for x = 2. Is it possible to define the value of f(x) in such a way that the redefined function becomes continuous at x = 2 ?

8. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous on R.

(i) $f(x) = \frac{x^2 - 2x - 8}{x + 2}$, $a = -2$

(ii) $f(x) = \frac{x - 7}{|x - 7|}$, $a = 7$

Practice Problems C

9. Test the function f(x) for continuity and indicate the kind of discontinuity, if any.

(i) $f(x) = x \sin(1/x)$

(ii) $f(x) = \sin(1/x)$,

(iii) $f(x) = \{x\}$,

(iv) $f(x) = \begin{cases} x^2 & \text{if } x \text{ is an irrational number} \\ 1 & \text{if } x \text{ is a rational number} \end{cases}$

(v) $f(x) = \frac{x^2 + 2}{x^3 + 1}$,

(vi) $f(x) = \tan^{-1}(1/x)$,

(vii) $f(x) = \frac{1}{1 - e^{x/(1-x)}}$,

(viii) $f(x) = \ln \frac{x^2}{(x+1)(x-3)}$

(ix) $f(x) = \begin{cases} x^2 & \text{for } 0 \le x < 1, \\ 2 - x & \text{for } 1 < x \le 2, \end{cases}$

(x) $f(x) = \begin{cases} \cos(\pi x/2), & |x| \le 1, \\ |x - 1|, & |x| > 1 \end{cases}$

10. What is the nature of the discontinuities at x = 0 of the functions:

(i) $y = \frac{\sin x}{x}$ (ii) $y = [x] + [-x]$

(iii) $y = \text{cosec } x$ (iv) $y = \sqrt{\left(\frac{1}{x}\right)}$,

(v) $y = \sqrt[3]{\left(\frac{1}{x}\right)}$ (vi) $y = \text{cosec}\,\frac{1}{x}$

(vii) $y = \frac{\sin(1/x)}{\sin(1/x)}$?

11. Can $f(x) = x(x^2 - 1)/|x^2 - 1|$ be extended to be continuous at x = 1 or –1?

12. Show that the only discontinuities a rational function can have are either removable or infinite. That is, if r(x) is a rational function that is not continuous at a, show that either a is a removable discontinuity or $\lim_{x \to a} |r(x)| = \infty$.

13. Let $f(x) = \lim_{n \to \infty} (1 + x)^n$. Comment on the continuity of f(x) at 0.

14. Investigate the following functions for continuity:

(i) $y = \frac{\sqrt{7 + x} - 3}{x^2 - 4}$

(ii) $y = (1 + x) \tan^{-1} \frac{1}{1 - x^2}$

(iii) $y = \frac{1}{1 + e^{\frac{1}{1-x}}}$

15. Investigate the following functions for continuity:

(i) $y = \lim_{n \to \infty} \frac{1}{1 + x^n}$ $(x \ge 0)$

(ii) $y = \lim_{n \to \infty} (x \tan^{-1} nx)$

2.4 ALGEBRA OF CONTINUOUS FUNCTIONS

It is easily deduced from the theorems on limits that the sum, product, difference or quotient of two functions which are continuous at a certain point are themselves continuous at that point (except that, in the case of the quotient, the denominator must not vanish at the point in question).

Further it is true that composition of a continuous function with a continuous funcion is a continuous function.

1. If f(x) and g(x) are continuous at x = a, then the following functions are also continuous at x = a.

(i) cf(x) is continuous at x = a, where c is any constant.

(ii) $f(x) \pm g(x)$ is continuous at x = a.

(iii) f(x). g(x) is continuous at x = a.

(iv) f(x)/g(x) is continuous at x = a, provided $g(a) \neq 0$.

Theorem If the functions f(x) and g(x) are continuous at a point x = a, then the sum h(x) = f(x) + g(x) is also continuous at the point x = a.

Proof Since f(x) and g(x) are continuous, we can write

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

By theorem on limits, we can write

$$\lim_{x\to a} h(x) = \lim_{x\to a} [f(x)+g(x)]$$

$$= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = f(a) + g(a) = h(a).$$

Thus, the sum h(x) = f(x) + g(x) is a continuous function.

Note, as a corollary, that the theorem holds true for any finite number of terms.

Theorem The sum of a finite number of functions continuous at a point is a continuous function at the point.

Proof Suppose we are given a definite number of functions $f_1, f_2, ..., f_n$ continuous at a point x = a. We have to prove that their sum $g = f_1 + f_2 + ... + f_n$ is a continuous function at that point. The functions $f_1, f_2, ..., f_n$ being continuous, we have

$$\lim_{x\to a} f_1(x) = f_1(a),\ \lim_{x\to a} f_2(x) = f_2(a), ...$$

$$\lim_{x\to a} f_n(x) = f_n(a),$$

By the theorem on the limit of a sum we write

$$\lim_{x\to a} g(x) = \lim_{x\to a}(f_1 + f_2 + ... + f_n)$$

$$= \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x) + ... + \lim_{x\to a} f_n(x)$$

$$= f_1(a) + f_2(a) + ... + f_n(a) = g(a).$$

Thus, $\lim_{x\to a} g(x) = g(a)$ which is what we wished to prove.

By the above theorem we at once recognize that the functions $y = x^2 + 3x + \sqrt{x}$,

$y = \sin^3 x - x \sin x - (x^4 - 1)\cos x$, and

$y = \sin x - 2x/(x^2 + 1)$ are continuous at every point in a domain common to all functions involved.

We can show that a polynomial is a continuous function. The polynomial function P : R ⟶ R is given by $P(x) = a_0 + a_1x + ... + a_nx^n$, where $a_0, a_1, ..., a_n \in R$. The functions $f_1, f_2, ..., f_n : R \to R$ defined by $f_1(x) = x, f_2(x) = x^2, ..., f_n(x) = x^n$ are continuous.

Hence, the functions $a_1f_1, a_2f_2, ..., a_nf_n$ are also continuous functions. Therefore, the function

$$P(x) = a_0 + a_1f_1 + ... + a_nf_n$$

is also continuous as the sum of continuous functions is a continuous function.

Theorem The product of a finite number of functions continuous at a point is a continuous function at that point.

Proof Let $g = f_1 . f_2 ..... f_n$ retaining the notation of the above theorem and using the theorem on the limit of a product, we get

$$\lim_{x\to a} g(x) = \lim_{x\to a}(f_1.f_2....f_n)$$

$$= \lim_{x\to a} f_1(x).\ \lim_{x\to a} f_2(x). .....\ .\lim_{x\to a} f_n(x)$$

$$= f_1(a) . f_2(a) ...... f_n(a) = g(a)$$

which is what we set out to prove.

Theorem The quotient of two functions continuous at a point x = a is a continuous function at the point x = a provided that the denominator does not turn into zero at the point.

Proof If $g = \frac{f_1}{f_2}$, then by the theorem on the limit of a quotient, we have

$$\lim_{x\to a} g(x) = \lim_{x\to a} \frac{f_1(x)}{f_2(x)} = \frac{\lim_{x\to a} f_1(x)}{\lim_{x\to a} f_2(x)} = \frac{f_1(a)}{f_2(a)} = g(a)$$

if $\lim_{x\to a} f_2(x) = f_2(a) \neq 0$. Therefore $g = \frac{f_1}{f_2}$ is a continuous function at the point x = a.

2. If f(x) is continuous at x = a and g(x) is discontinuous at x = a, then we have the following results.

(i) Both the functions f(x) + g(x) and f(x) – g(x) are discontinuous at x = a.

For example, consider, f(x) = x and g(x) = {x}. Here f(x) is continuous at x = 0 and g(x) is discontinuous at x = 0. Both the sum function x + {x} and the difference function x – {x} are discontinuous at x = 0.

(ii) f(x). g(x) is not necessarily discontinuous at x = a. We need to find the result by getting the limit of the product f(x). g(x) and comparing it with f(a). g(a).

For example, consider, $f(x) = x^3$ and g(x) = sgn(x). Here f(x) is continuous at x = 0 and g(x) is discontinuous at x = 0. But the product function

$$f(x)g(x) = \begin{cases} x^3, & x > 0 \\ 0, & x = 0 \\ -x^3, & x < 0 \end{cases}$$ is continuous at x = 0

As another example, the product of the functions

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous at x = 0 even when f(x) is continuous at x = 0 and g(x) is discontinuous at x = 0.

However, the product of the functions f(x) = x and g(x) = [x] is discontinuous at x = 1.

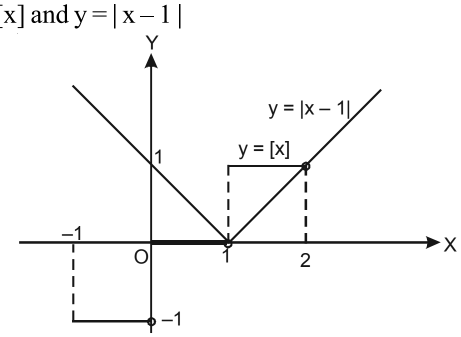
(iii) f(x)/g(x) is not necessarily discontinuous at x = a. Here also we need to work on the function f(x)/g(x) to get the result.

Let $f(x) = x(x^2 - 1)$ and $g(x) = \begin{cases} x+1, & x \ge 0 \\ x-1, & x < 0 \end{cases}$.

Here f(x) is continuous at x = 0 and g(x) is discontinuous at x = 0. We can check that f(x)/g(x) is continuous at x = 0.

Example 1. Discuss the continuity of f(x) = [x] + |x – 1|.

Solution Let us draw the graphs of the functions y = [x] and y = | x – 1 |



It is a clear from the figure that f(x) = [x] is discontinuous at all integral points and g(x) = |x – 1| is continuous for all x ∈ R. The sum of a discontinuous and a continuous function is discontinuous. Hence f(x) + g(x) is discontinuous at all integral points.

3. If f(x) and g(x) both are discontinuous at x = a, then we have the following results.

(i) The functions f(x) + g(x) and f(x) – g(x) are not necessarily discontinuous at x = a. However, atmost one of f(x) + g(x) or f(x) – g(x) can be continuous at x = a. That is, both of them cannot be continuous simultaneously at x = a. We have the following reason.

Let us assume that both f(x) + g(x) and f(x) – g(x) are continuous. Then the sum of functions (f(x) + g(x)) + (f(x) – g(x)) = 2f(x) must be continuous at x = a, which is wrong as it is given that f(x) is discontinuous at x = a. Hence our assumption is wrong. So, both the functions cannot be continuous simultaneously at x = a.

For example, consider, f(x) = [x] and g(x) = {x}. Here both f(x) and g(x) are discontinuous at x=0

The sum function [x] + {x} being equal to x is continuous at x = 0. The difference function [x] – {x} however is discontinuous at x = 0.

But this does not mean that one of the functions f(x) + g(x) or f(x) – g(x) must be continuous. We can have both the functions discontinuous. For example,

if f(x) = 2[x] and g(x) = {x}, then both the functions f(x) + g(x) or f(x) – g(x) are discontinuous simultaneously at x = 0 .

(ii) f(x). g(x) is not necessarily discontinuous at x = a. We need to find the result by getting the limit of the product f(x). g(x) and comparing it with f(a). g(a).

For example, consider, f(x) = [x] and g(x) = [–x]. Here both f(x) and g(x) are discontinuous at x = 0 but, the product function [x]·[–x] is continuous at x = 0.

Further, f(x) = [x] and g(x) = {x} are both discontinuous at x = 0 and, the product function [x]·{x} is discontinuous at x = 0.

Hence, we cannot comment in advance, when both the functions are discontinuous at x = a.

(iii) f(x)/g(x) is not necessarily discontinuous at x = a. Here also we need to work on the function f(x)/g(x) to get the result.

Let $f(x) = \begin{cases} x^2 - 1, & x \ge 0 \\ x+1, & x < 0 \end{cases}$ and $g(x) = \begin{cases} x+1, & x \ge 0 \\ x-1, & x < 0 \end{cases}$.

Here both f(x) and g(x) are discontinuous at x = 0. But we find that f(x)/g(x) is continuous at x = 0.

Example 2. If f(x) = [sin(x–1)] – {sin(x–1)} then comment on continuity of f(x) at $x = \frac{\pi}{2} + 1$.

Solution $f(x) = [\sin(x-1)] - \{\sin(x-1)\}$ Let $g(x) = [\sin(x-1)] + \{\sin(x-1\} = \sin(x-1)$

which is obviously continuous at $x = \frac{\pi}{2} + 1$

Here $[\sin(x-1)]$ and $\{\sin(x-1)\}$ are both discontinuous at $x = \frac{\pi}{2} + 1$.

$\therefore$ Atmost one of f(x) or g(x) can be continuous at $x = \frac{\pi}{2} + 1$.

As g(x) is continuous at $x = \frac{\pi}{2} + 1$, therefore, f(x) must be discontinuous.

Continuity of Composite Functions

Theorem A function composed of a finite number of continuous functions is a continuous function.

It is sufficient to prove this assertion for a composite function formed by two continuous functions because after that it can be extended, consecutively, to an arbitary number of constituent functions.

Theorem If f(x) is continuous at x = a and g(x) is continuous at x = f(a) then the composite function (gof)(x) is continuous at x = a.

Proof Let y = g(u), u = f(x) and y = g(f(x)) = F(x) where f(x) is continuous for x = a and g(u) is continuous for u = b = f(a). We have to prove that

y = F(x)) is continuous at the point a.

Indeed, let $x \to a$. The continuity of the function

u = f(x) implies that $\lim_{x\to a} f(x) = f(a) = b$,

that is $u \to b$. The function g(u) being continuous at the point b. we have $\lim_{u\to b} g(u) = g(b)$.

Now, since u = f(x), we can rewrite the last relation in the form $\lim_{x\to a} g(f(x)) = g(f(b))$ or, equivalently,

$\lim_{x\to a} F(x) = F(a)$ which is what we wished to prove.

This theorem is also named as the chain rule for continuity.

For example, f(x) = sin x is continuous at $x = \frac{\pi}{2}$ and

$g(x) = \begin{cases} x^2 - 1, & x \le 1 \\ x - 1, & x > 1 \end{cases}$ is continuous at $x = f(\frac{\pi}{2}) = 1$.

Hence the composite function (gof)(x) is continuous at $x = \frac{\pi}{2}$.

Note:

1. Let a function f(x) be continuous at all points in the interval [a, b], and let its range be the interval [A, B] and further the function g(x) be continuous in the interval [A, B], then the composite function (gof)(x) is continuous in the interval [a, b].
2. If the function f is continuous everywhere and the function g is continuous everywhere, then the composition gof is continuous everywhere.
3. All polynomials, trigonometric functions, inverse trigonometric functions, exponential and logarithmic functions are continuous at all points in their domains.
4. If f(x) is continuous, then | f(x) | is also continuous. For example, $f(x) = \frac{x \sin x}{x^2 + 2}$ and $g(x) = |x|$ are continuous for all x. Hence, the composite function $(gof)(x) = \left|\frac{x \sin x}{x^2 + 2}\right|$ is also continuous for all x.

Example 3. Check the continuity of

$$f(x) = \frac{x^3 \cos x + x^2 \sin x}{\cos(1/\sin x)}.$$

Solution The numerator is continuous for all x. As far as the denominator is concerned, according to the theorem on continuity of a composite function, it is continuous at points where the function u = 1/sin x is continuous, since the function cos u is continuous everywhere.

Hence the denominator is continuous everywhere, except at the point $x = k\pi$ (k an integer). Besides, we must exclude the points at which cos (1/sin x) = 0. i.e. the points at which $1/\sin x = (2p + 1)\pi/2$ ($p \in I$), or $\sin x = 2/[2p + 1)\pi]$. Thus, the function f(x) is continuous everywhere except at the points

$x = k\pi$ and $x = (-1)^n \sin^{-1} \frac{2}{(2p+1)\pi} + n\pi$,

where $k, p, n \in I$.

Example 4. What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

Solution Because the natural domain of this function is the closed interval [–3, 3], we will need to investigate the continuity of f on the open interval

(–3, 3) and at the two endpoints. If c is any number in the interval (–3, 3), then $\lim_{x\to c} f(x) = \lim_{x\to c}\sqrt{9-x^2}$

$$= \sqrt{\lim_{x\to c}(9-x^2)} = \sqrt{9-c^2} = f(c)$$

which proves f is continuous at each number in the interval (–3, 3). The function f is also continuous at the endpoints since

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}\sqrt{9-x^2} = \sqrt{\lim_{x\to 3^-}(9-x^2)} = 0 = f(3);$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+}\sqrt{9-x^2}$$

$$= \sqrt{\lim_{x\to -3^+}(9-x^2)} = 0 = f(-3).$$

Thus, f is continuous on the closed interval [–3, 3].

Note: The nth-root function $f(x) = \sqrt[n]{x}$ is continuous everywhere if n is odd and it is continuous for $x \geq 0$ if n is even.

We may combine this result with the previous theorem. Then we see that a root of a continuous function is continuous, wherever it is defined. That is, the composition

$$h(x) = \sqrt[n]{g(x)} = [g(x)]^{1/n}$$

of $f(x) = \sqrt[n]{x}$ and the function g(x) is continuous at a assuming that $g(a) \geq 0$ if n is even (so that $\sqrt[n]{g(a)}$ is defined).

For example, the above theorem tells us that composite functions like $\sqrt{\sin x}$, $\sqrt{x}$, $(\sqrt{x}+1)^3$, $\sqrt{\cos^2 x + 3}$, are continuous at all points at which the functions are defined, because polynomials and trigonometric functions like sine and cosine are continuous at every point, while root functions are continuous on their proper domains.

Example 5. Show that the function

$$f(x) = \left(\frac{x-7}{x^2+2x+2}\right)^{2/3}$$ is continuous everywhere.

Solution Note first that the denominator

$$x^2+2x+2 = (x+1)^2+1 \text{ is never zero.}$$

Hence the rational function

$$r(x) = \frac{x-7}{x^2+2x+2}$$

is defined and continuous everywhere. It then follows from the continuity of the cube root function that

$f(x) = [r(x)]^{2/3} = \sqrt[3]{[r(x)]^2}$ is continuous everywhere.

As was mentioned earlier, all the basic elementary functions are continuous in the intervals where they are defined and therefore the theorems proved here imply that every elementary function is continuous in those intervals where it is defined.

An elementary function can only be discontinuous at those points where some of the constituent functions it is formed of are not defined or where the denominators of some fractions involved vanish.

For instance, the function $y = \dfrac{x}{x^2-4}$ is discontinuous at the points x = ± 2 and continuous at all the other points.

The function $y = x^2 \tan x$ is discontinuous at the points $x = (2k+1)\dfrac{\pi}{2}$, $k \in I$.

Example 6. Find the points of discontinuity of

$$y = \frac{1}{u^2+u-2}, \text{ where } u = \frac{1}{x-1}.$$

Solution The function $u = f(x) = \dfrac{1}{x-1}$ is discontinuous at the point x = 1. ...(1)

The function $y = g(u) = \dfrac{1}{u^2+u-2} = \dfrac{1}{(u+2)(u-1)}$ is discontinuous at u = –2 and u = 1.

When u = –2, $\dfrac{1}{x-1} = -2$

$\Rightarrow \quad x-1 = -\dfrac{1}{2} \Rightarrow x = 1/2$ ...(2)

When u = 1, $\dfrac{1}{x-1} = 1$

$\Rightarrow \quad x-1 = 1 \quad \Rightarrow \quad x = 2$ ...(3)

Hence, the composite function y = g(f(x)) is discontinuous at three points $x = \dfrac{1}{2}$, 1, 2.

Example 7. If $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{1}{x-2}$, then discuss the continuity of f(x), g(x) and (fog)(x).

Solution $f(x) = \dfrac{x+1}{x-1}$

f(x) is a rational function it must be continuous in its domain. f is not defined at x = 1

∴ f is discontinuous at x = 1

$$g(x) = \frac{1}{x-2}$$

g(x) is also a rational function. It must be continuous in its domain and it is not defined at x = 2

∴ g is discontinuous at x = 2

Now fog(x) may be discontinuous at

(i) x = 2 (point of discontinuity of g(x))

(ii) g(x) = 1 (when g(x) = point of discontinuity of f(x)).

If g(x) = 1 then $\frac{1}{x-2} = 1 \Rightarrow \quad x = 3.$

∴ The discontinuity of fog(x) should be checked at x = 2 and x = 3

At x = 2, $\quad fog(x) = \dfrac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}$

We see that fog (2) is not defined.

$$\lim_{x\to 2} fog(x) = \lim_{x\to 2} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \lim_{x\to 2} \frac{1+x-2}{1-x+2} = 1.$$

∴ fog (x) is discontinuous at x = 2 and it has a removable discontinuity at x = 2.

At x = 3, fog (3) is not defined.

$$\lim_{x\to 3^+} fog(x) = \lim_{x\to 3^+} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \infty$$

$$\lim_{x\to 3^-} fog(x) = \lim_{x\to 3^-} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = -\infty$$

∴ fog (x) is discontinuous at x = 3 and it has an infinite irremovable discontinuity there.

Example 8. Given the function f(x) = 1/(1 – x). Find the points of discontinuity of the composite function y = f(f(f(x))).

Solution The point x = 1 is a discontinuity of the function $v = f(x) = \frac{1}{1-x}$. If x ≠ 1, then

$$u = f(f(x)) = \frac{1}{1-1/(1-x)} = \frac{x-1}{x}.$$

Hence, the point x = 0 is a discontinuity of the function u = f(f(x)). If x ≠ 0, x ≠ 1, then

$y = f(f(f(x))) = \dfrac{1}{1-(x-1)/x} = x$ is continuous everywhere. Thus, the points of discontinuity of this composite function are x = 0, x = 1, both of them being removable.

Example 9. If f(x) = sgn(2sinx + a) is continuous for all x, then find the possible values of a.

Solution Since f(x) = sgn(2sinx + a) is continuous for all x, we should have 2sinx + a ≠ 0 for any real x.

⇒ sin x ≠ –a/2

⇒ |a/2| > 1

⇒ a < –2 or a > 2.

Continuity of the Inverse of a Continuous Invertible Function

Theorem If the function y = f(x) is defined, continuous and strictly monotonic on the interval I, then there exist the inverse function $y = f^{-1}(x)$ defined, continuous and also strictly monotonic in the range of the function y = f(x).

Theorem Assume f is strictly increasing and continuous on an interval [a, b]. Let c = f(a) and d = f(b) and let g be the inverse of f. Then

(i) g is strictly increasing on [c, d], and

(ii) g is continuous on [c, d]

There is a coresponding theorem for decreasing functions. That is, the inverse of a strictly decreasing continuous function f is strictly decreasing and continuous.

For example, we prove that the function $y = \sqrt[3]{x}$ is continuous ∀ x ∈ R, considering it as the inverse of $y = x^3$. The function $y = x^3$ is continuous ∀ x ∈ R, and its range is y ∈ R. Also, it is strictly increasing and hence invertible. Hence, its inverse function,

$y = \sqrt[3]{x}$, x ∈ R, is continuous ∀ x ∈ R.

Continuity of Integrals

Assume f is integrable on [a, x] for every x in [a, b] and let $A(x) = \int_a^x f(t)dt$. Then the integral A is continuous at each point of [a, b]. (At each endpoint we have one-sided continuity)

Proof We choose a point c in [a, b]. Now we prove that $A(x) \to A(c)$ as $x \to c$. We have

$$A(x) - A(c) = \int_c^x f(t)dt$$

We estimate the size of this integral. Since f is bounded on [a, b], there is a constant M > 0 such that $-M(x-c) \le A(x) - A(c) \le M(x-c)$.

If x < c, we obtain the same inequalities with x – c replaced by c – x. Therefore, in either case we can let $x \to c$ and apply the Sandwich theorem to find that $A(x) \to A(c)$. This proves the theorem. If c is an endpoint of [a, b], we must let $x \to c$ from inside the interval, so the limits are one-sided.

Note: If f is continuous on [a, b], then it is integrable on the interval. Hence, the integral of a continuous function is continuous. For example, $\int_1^x \frac{\sin t}{t}dt$ is continuous in the interval $[1, \infty)$.

Concept Problems D

1. Let $f(x) = \begin{cases} 1 & x \ge 0 \\ -1 & x < 0 \end{cases}$ and

$g(x) = \begin{cases} -1 & x \ge 0 \\ 1 & x < 0 \end{cases}$

Show that f + g is continuous at x = 0 even though f and g are both discontinuous there.

2. Will the sum of two functions f(x) + g(x) be necessarily discontinuous at a given point x_0 if :

(i) The function f(x) is continuous and the function g(x) is discontinuous at $x = x_0$,

(ii) Both functions are discontinuous and the function at $x = x_0$?

3. Is the product of the two functions f(x) g(x) necessarily discontinuous at a given point x_0 if :

(i) The function f(x) is continuous and the function g(x) is discontinuous at this point;

(ii) Both functions f(x) and g(x) are discontinuous at $x = x_0$?

4. Prove that if the function f(x) is continuous and non-negative in the interval (a, b), then the function $F(x) = \sqrt{f(x)}$ is likewise continuous in this interval.

5. Prove that $f(x) = 1/\sqrt{x^4 + 7x^2 + 1}$ is continuous everywhere.

6. Suppose that the function f is continuous everywhere and that the composition f(g(x)) is continuous at x = a. Does it follow that g(x) is continuous at a ?

7. Can one assert that the square of a discontinuous function is also a discontinuous function? Give an example of a function discontinuous everywhere whose square is a continuous function.

8. Give an example of a function f such that f is not continuous but |f| is continuous. Show that f^2 can be continuous when f is not.

Practice Problems D

9. Let $f(x) = \begin{cases} x & \text{if } x \ne 0 \\ 2 & \text{if } x = 0 \end{cases}$ and

$g(x) = \begin{cases} 3x & \text{if } x \ne 0 \\ -2 & \text{if } x = 0 \end{cases}$

Show that f + g is continuous at x = 0 even though f and g are both disontinuous there.

10. Let $f(x) = x^2$ and

$g(x) = \begin{cases} \cos\frac{\pi}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$

Show that f.g is continuous at x = 0.

11. Let f and g be two functions defined as follows:

$f(x) = \frac{x + |x|}{2}$ for all x, $g(x) = \begin{cases} x & \text{for } x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases}$

Find a formula for the composite function h(x) = f[g(x)]. For what values of x is h continuous?

12. Let f and g be two functions defined as follows:

$f(x) = \begin{cases} 1 + x^3 & , \quad x < 0 \\ x^2 - 1 & , \quad x \ge 0 \end{cases}$;

$$g(x)=\begin{cases}(x-1)^{\frac{1}{3}} & , \quad x<0 \\ (x+1)^{\frac{1}{2}} & , \quad x\ge 0\end{cases}$$

Comment the continuity of gof(x)

13. Let f and g be two functions defined as follows :

$$f(x)=\begin{cases}1 & \text{if} \quad |x|\le 1 \\ 0 & \text{if} \quad |x|>1\end{cases}, \quad g(x)=\begin{cases}2-x^2 & \text{if} \quad |x|\le 2 \\ 2 & \text{if} \quad |x|>2\end{cases}$$

Find a formula for the composite function $h(x)=f[g(x)]$. For what values of x is h continuous?

14. Discuss the continuity of the composite function

$h(x)=f(g(x))$ where $f(x)=\dfrac{1}{(x-6)}$, $g(x)=x^2+5$.

15. If $f(x)=\dfrac{1}{1-x}$. Find the points of discontinuity of the function $y=f(f(f(x)))$.

16. Let $f(x)=\begin{cases}1+x & , \quad 0\le x\le 2 \\ 3-x & , \quad 2<x\le 3\end{cases}$.

Determine the form of $g(x)=f(f(x))$ and hence find the point of discontinuity of g, if any.

17. If $f(x)=-1+|x-1|$, $-1\le x\le 3$ and $g(x)=2-|x+1|$, $-2\le x\le 2$, then discuss the continuity of $f(g(x))$.

18. Let $f(x)=\begin{cases}x-1, & -1\le x<0 \\ x^2, & 0\le x\le 1\end{cases}$ and $g(x)=\sin x$.

Further let $h(x)=f(|g(x)|)+|f(g(x))|$. Discuss the continuity of h(x) in [–1, 1].

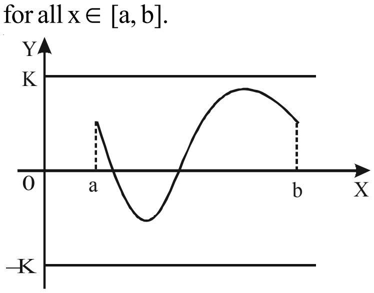
2.5 PROPERTIES OF FUNCTIONS CONTINUOUS ON A CLOSED INTERVAL

Functions continuous on a closed interval possess d in the following theorems.

Boundedness Theorem

If a function f(x) is continuous at every point of a closed interval [a, b], then the function f(x) is bounded on this interval.

Note that the boundedness of a function on the interval [a, b] means that there is a number $K>0$ such that $|f(x)|<K$ for all $x\in[a, b]$.



Represented in the figure is the graph of a continuous function f on a closed interval [a, b]. Obviously, there exists a number $K>0$ such that the graph is located between the straight lines $y=K$ and $y=-K$.

Note that if a function is continuous on an open interval (a, b) or on a half-open interval [a, b) or (a, b], then it is not necessarily bounded on such an interval. For instance, the function $y=1/x$ is continuous but not bounded in the interval (0, 1]. There is no contradiction with the theorem since the function in question is not continuous on a closed interval but only in a half-open one.

If the continuity condition is not take into account, then the assertion that the function f(x) is bounded may not be true. For instance, the function

$$y=\begin{cases}1/x & \text{for} \quad 0<x\le 1 \\ 0 & \text{for} \quad x=0\end{cases}$$

is defined on the closed interval [0, 1], but is not bounded on this interval since it is discontinuous at $x=0$.

Weierstrass Theorem (Extreme Value Theorem)

If a function is continuous in a closed interval there exists atleast one point at which the function assumes the greatest value and atleast one point at which it assumes the least value on that interval.

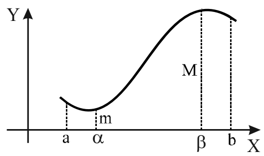
If a function f is continuous on a closed interval [a, b], then there exist its minimum and maximum values on [a, b], i.e. there exist points $\alpha, \beta, \in[a, b]$ such that $f(\alpha)\le f(x)\le f(\beta)$ for all $x\in[a, b]$. In other words,

$$\min_{x\in[a,b]} f(x)=f(\alpha), \qquad \max_{x\in[a,b]} f(x)=f(\beta).$$

In other words, if $m=\min\{f(x) : a\le x\le b\}$ and $M=\max\{f(x) : a\le x\le b\}$, then there are two points α, $\beta\in[a, b]$ such that $f(\alpha)=m$ and $f(\beta)=M$. Note that these points α and β need not be unique.

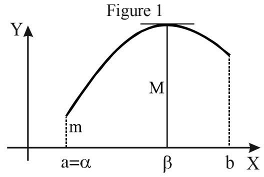
The continuous function $y=f(x)$ represented in Figure 1 attains its minimum on [a, b] at the point $x=\alpha$ and maximum at the point $x=\beta$. In this case both points α and β belong to the interval (a, b).

The continuous function y = f(x) represented in Figure 2 reaches its minimum on [a, b] at its left end point and maximum at a certain interior point β of this interval.



Minimum value 'm' occurs at x = α and maximum value M occurs at x = β, α, β ∈ (a, b)

Figure 1



Minimum value 'm' occurs at the end point x = a and the maximum value M occurs inside the interval

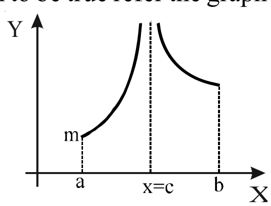
Figure 2

The value (α) and f(β), whose existence is affirmed by the theorem, are respectively the absolute minimum and absolute maximum of the function on the interval [a, b].

The extreme value theorem guarantees that the maximum and the minimum exists, but does not tell us how to find them. The problem of finding them is discussed in the chapter of maxima – minima.

If the function f(x) is continuous in the open interval (a, b) or in the half-open interval (a, b], then the function may not attain its least or greatest value. Further, if f(x) is a discontinuous function, then the theorem may not hold true.

To see that continuity is necessary for the extreme value theorem to be true refer the graph shown below.



There is a discontinuity at x = c in the interval. The function has a minimum value at the left end point x = a and f has no maximum value.

Also, the function

$$f(x) = \begin{cases} x & \text{for} \quad 0 \le x < 1, \\ 0 & \text{for} \quad x = 1 \end{cases}$$

which is continuous on an open interval [0, 1) has no absolute maximum. As $x \to 1$ from the left, $f(x) \to 1$, but f(x) does not attain the value 1. Incidentally, f takes on its minimum value 0 on the interval [0,1] at two places, at x = 0 and at x = 1

If we consider the function y = x in the interval 0 < x < 1, there is no least and no greatest values among them. There is no extreme left point, since no matter what point $x = x_1$ we take there will be a point to the left of it, for instance, the point $\frac{x_1}{2}$. Likewise there is no extreme right point; consequently, there is no least and no greatest value of the function y = x, 0 < x < 1.

Consider the function $f(x) = \tan^{-1}x$ for $x \ge 0$.

It is obvious that $\lim_{x\to\infty} \tan^{-1} x = \pi/2$. But there is no x for which the function $\tan^{-1}x$ takes on the value π/2, and it does not attain maximum on $x \ge 0$. In this case the conditions of the theorem are not fulfilled : here the domain of the function [0, ∞) is unbounded.

If a function f is discontinuous, then it may have both a maximum and a minimum value, but this is not always true.

For example, the function $f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 3 - x, & 2 < x \le 4 \end{cases}$

has a maximum at x = 2 and a minimum at x = 4, even if it is discontinuous of x = 2. This can be concluded from the graph of y = f(x).

Example 1. Let $f(x) = 1/(1-x^2)$ in the open interval (–1, 1). Show that f does not have a maximum value in this interval.

Solution For x near 1, f(x) gets arbitrarily large since the denominator $1-x^2$ is close to 0. The graph of f, for x ∈ (–1,1), shows that the function is continuous throughout the open interval (–1, 1), but there is no number c in (–1,1) at which f has a maximum value. However, f has a minimum value, f(0) = 1.

Sign Preserving Property of Continuous Functions

If f is continuous at c and f(c) ≠ 0, then there exists an interval (c – δ, c + δ) around c such that f(x) has the sign of f(c) for every value of x in this interval.

Its truth is obvious if we understand that a continuous function does not undergo sudden changes so that if f(c) is positive for the value c of x and also f is continuous at c, it cannot suddenly become negative or zero and must, therefore, remain positive for values of x in a certain neighbourhood of c.

If $\lim_{x\to c} f(x) = b$ and b > 0, then there exists a deleted neighbourhood D of c such that f(x) > 0 for every x in D.

Similarly, if b < 0 there exists a D such that f(x) < 0 for every x in D.

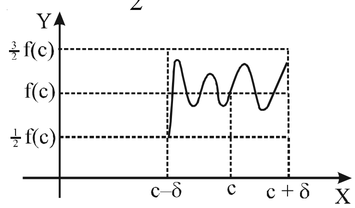
Proof Suppose f(c) > 0. By continuity, for every $\in > 0$ there is a $\delta > 0$ such that

$$f(c) - \in < f(x) < f(c) + \in \qquad ...(1)$$

whenever $c - \delta < x < c + \delta$.

If we take the δ corresponding to $\in = f(c)/2$ (this $\in$ is positive), then (1) becomes

$$\frac{1}{2}f(c) < f(x) < \frac{3}{2}f(c),\ c - \delta < x < c + \delta.$$



Here f(x) > 0 for x near c because f(c) > 0.

Therefore f(x) > 0 in this interval, and hence f(x) and f(c) have the same sign. If f(c) < 0, we take the δ corresponding to $\in = -\frac{1}{2}f(c)$ and arrive at the same conclusion.

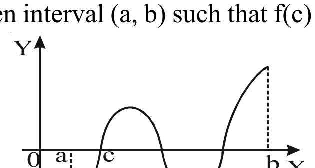
Corollary. If f(x) is continuous at x = c, and f(x) vanishes for values of x as near as we please, or assumes, for values of x as near to c as we please, both positive and negative values, then f(c) = 0.

This is an obvious corollary to sign preserving property. If f(c) is not zero, it must be positive or negative; and if it were, for example, positive, f(x) would be positive for all values of x sufficiently near to c, which contradicts the hypotheses of the theorem.

Note: If there is one-sided continuity at c, then there is a corresponding one-sided interval $[c, c + \delta)$ or $(c - \delta, c]$ in which f has the same sign as f(c).

Bolzano's Theorem

If a function f is continuous on a closed interval [a, b] and the numbers f(a) and f(b) are different from zero and have opposite signs, then there is atleast one point c on the open interval (a, b) such that f(c) = 0.



The function whose graph is depicted in the above figure satisfies the conditions of Bolzano's theorem. It is continuous on [a, b] and f(a) < 0, f (b) > 0. Geometrically, it is obvious that the graph must intersect the x-axis atleast at one point $c \in (a, b)$. This is just what is stated by the theorem.

In other words, if f is continuous in [a, b] and f(a) and f(b) have opposite signs, then there is atleast one value of x for which f(x) vanishes in the interval (a, b).

Proof To fix the ideas, suppose that f(a) < 0 and f(b) > 0. Since f(x) is continuous, it will be negative in the neighbourhood of a and positive in the neighbourhood of b. The set of values of x between a and b which make f(x) positive is bounded below by a, and hence possesses an exact lower bound k: clearly a < k < b.

From the definition of the lower bound, the values of f(x) must be negative or zero in $a \leq x < k$. Since f(x) is continuous when x = k,

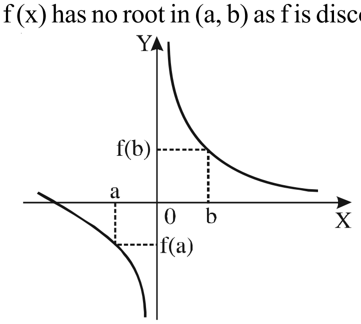
$$\lim_{x \to k^-} f(x) = f(k).$$

Hence f(k) is also negative or zero. We shall show that f(k) cannot be negative; for if f(k)= – c, where c is positive, then there exists a positive number δ such that

$$|f(x) - f(k)| < c \text{ when } |x - k| \leq \delta,$$

since f(x) is continuous when x = k. The function f(x) would when be negative for whose values of x in (a, b) which lie between k and $k + \delta$, which contradicts the fact that k is the lower bound of the set of values of x between a and b which make f(x) positive. It follows that f(k) = 0, and the theorem is therefore proved.

In the following figure, f (a) and f (b) are of opposite signs but f (x) has no root in (a, b) as f is discontinuous.



Hence, if a function is discontinuous in the interval [a, b] and f(a) and f(b) have opposite signs then f(x) may or maynot have a root in (a, b).

Example 2. The equation $x^3 - x - 2 = 0$ may have a solution somewhere between x = 1 and x = 2. Apply the Bolzano's theorem to show that this is true.

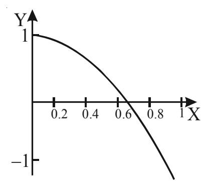
Solution The function $f(x) = x^3 - x - 2$ is continuous on [1, 2] because it is a polynomial. $f(1) = -2$ and $f(2) = 4$. Since, they have opposite signs, Bolzano's theorem implies that the function vanishes at some point in (1, 2). In particular, there exists atleast one c in (1, 2) such that $f(c) = c^3 - c - 2 = 0$.
So x = c is a solution of the equation $x^3 - x - 2 = 0$ which lies in (1, 2).

Example 3. What can be said about the roots of $f(x) = \cos x - 2x^2$ in the interval $0 \le x \le 1$?

Solution $f(x) = \cos x - 2x^2$ in the interval $0 \le x \le 1$. Since f(x) is a difference of the two continuous functions, it is continuous. We find values of f at different values of x as shown in the table.

TABLE

x	f(x)
0	1.00
0.2	0.90
0.4	0.60
0.6	0.11
0.8	–0.58
1.0	–1.46



Roots occur where the graph of a function crosses the x-axis

We conclude that f(x) has atleast one root in the interval $0.6 < x < 0.8$, since f(x) changes sign from positive to negative on that interval. The graph of f(x) suggests that there is only one root in the interval $0 \le x \le 1$, but we cannot find the root from the graph or the table of values.

Example 4. Given that $a > b > c > d$ then prove that the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ will have two real and distinct roots.

Solution $(x - a)(x - c) + 2(x - b)(x - d) = 0$

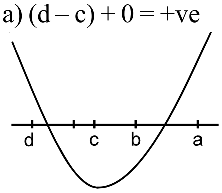
Let $f(x) = (x - a)(x - c) + 2(x - b)(x - d)$

$f(a) = (a - a)(a - c) + 2(a - b)(a - d) = +\text{ve}$

$f(b) = (b - a)(b - c) + 0 = -\text{ve}$

$f(c) = 0 + 2(c - b)(c - d) = -\text{ve}$

$f(d) = (d - a)(d - c) + 0 = +\text{ve}$



Hence $(x - a)(x - c) + 2(x - b)(x - d) = 0$ has atleast one real root in (d, c) and atleast one real root in (b, a). Since, a quadratic equation can have atmost two real roots, exactly one real root lies in each of the two intervals. Thus the roots are real and distinct.

Example 5. Show that $e^{-x} \sin x = \ln x$ has atleast one solution on the interval [1, 2].

Solution Notice that the function
$f(x) = e^{-x} \sin x - \ln x$ is continuous on [1, 2].
We find that

$f(1) = e^{-1} \sin 1 - \ln 1 \approx 0.31 > 0$ and

$f(2) = e^{-2} \sin 2 - \ln 2 \approx -0.57 < 0$

Therefore, by Bolzano theorem there is atleast one number c on (1, 2) for whieh f(c) = 0, and it follows that $e^{-c} \sin c = \ln c$.

Example 6. Given a function on the interval [–2, 2]

$$f(x) = \begin{cases} x^2 + 2 & \text{if } -2 \le x < 0, \\ -(x^2 + 2) & \text{if } 0 \le x \le 2 \end{cases}$$

is there a point on this closed interval at which f(x) = 0?

Solution At the endpoints of the interval [–2, 2] the given function has different signs :

$$f(-2) = 6;\ f(2) = -6.$$

But it is easy to notice that it does not become zero at any point of the interval [–2, 2]. Indeed, $x^2 + 2 > 0$ and $-(x^2 + 2) < 0$ at any x; this is due to the fact that f(x) has a discontinuity at the point x = 0.

Example 7. Let f be a continuous function defined from [0, 1] to [0, 1] with range [0, 1]. Show that is some 'c' in [0, 1] such that $f(c) = 1 - c$.

Solution Consider $g(x) = f(x) - 1 + x$

$g(0) = f(0) - 1 \le 0$ (as $f(0) \le 1$)

$g(1) = f(1) \ge 0$ (as $f(1) \in [0, 1]$)

There are three cases:

Case I: $g(0) = 0$

This happens when f(0) = 1. In such a case c = 0, which lies in [0, 1].

Case II: $g(1) = 0$

This happens when f(1) = 0. In such a case c = 1, which lies in [0, 1].

Case III: g (0) and g (1) are of opposite signs

By Bolzano's theorem there exists atleast one $c \in (0, 1)$ such that $g(c) = 0$.

$\Rightarrow g(c) = f(c) - 1 + c = 0 \quad \Rightarrow f(c) = 1 - c$.

Combining the three cases, there exists some c in [0, 1] such that $f(c) = 1 - c$.

Example 8. Let $f : [0, 2] \to R$ be continuous and $f(0) = f(2)$. Prove that there exists x_1 and x_2 in (0, 2) such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$

Solution Consider the continuous function g on $[0, 2]$ defined as $g(x) = f(x+1) - f(x)$ $(x_2 = x_1 + 1)$

Now, $g(0) = f(1) - f(0) = f(1) - f(2)$ ...(1)

$g(1) = f(2) - f(1) = f(2) - f(1)$ ...(2)

Thus, g (0) and g (1) are of opposite signs.

Hence by Bolzano's theorem there exists some $c \in (0, 1)$ where $g(c) = 0$,

i.e. $f(c+1) = f(c)$ $\{c + 1 \in (1, 2) \text{ as } c \in (0, 1)\}$

Putting $c = x_1$; $c + 1 = x_2$, obviously $x_1, x_2 \in (0, 2)$ we have $f(x_2) = f(x_1)$ where $x_2 - x_1 = 1$.

Example 9. Prove that the function

$$f(x) = a\sqrt{x-1} + b\sqrt{2x-1} - \sqrt{2x^2 - 3x + 1}$$

where a + 2b = 2 and $a, b \in R$ always has a root in $(1, 5)$ $\forall b \in R$.

Solution There are three cases:

Case I: Let $b > 0$, then $f(1) = b > 0$

and $f(5) = 2a + 3b - 6 = 2(a + 2b) - b - 6$

$= 4 - b - 6 = -(2 + b) < 0$

Hence by Bolzano's theorem, there exists some $c \in (1, 5)$ such that $f(c) = 0$.

Case II: If $b = 0$ then $a = 2$.

$$f(x) = 2\sqrt{x-1} - \sqrt{2x^2 - 3x + 1} = 0$$

$$\Rightarrow \quad 4(x-1) = 2x^2 - 3x + 1 = (2x-1)(x-1)$$

$$\Rightarrow \quad (x-1)(2x-5) = 0 \Rightarrow x = \frac{5}{2}$$

Hence $f(x) = 0$ if $x = \frac{5}{2}$ which lies in $(1, 5)$.

Case III: If $b < 0$, $f(1) = b < 0$ and

$$f(2) = a + b\sqrt{3} - \sqrt{3}$$

$$= (a + 2b) + (\sqrt{3} - 2)b - \sqrt{3}$$

$$= (2 - \sqrt{3}) - (2 - \sqrt{3})\, b$$

$$= (2 - \sqrt{3})(1 - b) > 0 \text{ (as } b < 0)$$

Thus, f(1) as f(2) have opposite signs

Hence there exists some $c \in (1, 2) \subset (1, 5)$ for which $f(c) = 0$.

Isolation of roots

A real number x_1 is said to be a real root of the equation $f(x) = 0$ if $f(x_1) = 0$. We say that a real root of an equation has been isolated if we exhibit an interval [a, b] containing this root.

If f is a continuous function in an interval I and $f(a)\, f(b) < 0$ for some $a, b \in I$, then by Bolzano's theorem there is a point c between a and b for which f(c) = 0. This is often used to locate the roots of equations of the form $f(x) = 0$.

For example, consider the equation $x^3 + x - 1 = 0$.

Let $f(x) = x^3 + x - 1$.

Note that f(0) = –1 whereas f(1) = 1. This shows that the above equation has a root between 0 and 1.

Now we try with 0.5, f(0.5) = –0.375. So there must be a root of the equation between 1 and 0.5. We again try 0.75, f(0.75) > 0, which means that the root is between 0.5 and 0.75.

So we may try 0.625, f(0.625) < 0. So the root is between 0.75 and 0.625.

Now if we take the approximate root to be .6875, then we are away from the exact root by atmost a distance of .0625. If we continue this process further, we shall get better and better approximations to the root of the equation.

Concept Problems E

1. The boundedness theorem tells us that every continuous real-function on the closed interval [0, 1] is bounded and attains its bounds. For each of the following intervals I give an example of a continuous function f from I to R which is unbounded, and an example of a function g from I to R which is continuous and bounded on I, but such that g does not have a maximum value on I.
   (i) $I = [0, 1)$; (ii) $I = (1, \infty)$; (iii) $I = [0, \infty)$.
2. Prove that the equations have a solution in the given intervals
   (i) $x^4 + 2x - 1 = 0$ on $[0, 1]$;
   (ii) $x^5 - 5x^3 + 3 = 0$ on $[-3, -2]$
3. Prove that the equation $\sin x - x \cos x = 0$ has a root between π and $\frac{3}{2}\pi$.
4. Let $f(x) = \tan x$. Although $f(\pi/4) = 1$ and $f(3\pi/4) = -1$, there is no x in the interval $(\pi/4, 3\pi/4)$ such that $f(x) = 0$. Explain why this does not contradict Bolzano's theorem.
5. With the aid of Bolzano's theorem, isolate the real roots of each of the following equations (each

has four real roots).

(i) $2x^4 - 14x^2 + 14x - 1 = 0$

(ii) $x^4 + 4x^3 + x^2 - 6x + 2 = 0$

6. Suppose that f is continuous on the interval [0, 1], that f(0) = 2, and that f has no zeros in the interval. Prove that f(x) > 0 for all x in [0, 1].

Practice Problems E

7. A function f from R to R is said to be periodic, and a real number T > 0 is said to be a period of f it, for all $x \in R$, $f(x + T) = f(x)$. Suppose that f is a continuous function from R to R and that f is periodic. Prove that f is bounded.

8. Does the function $2^x - x^3 + x^5$ have (a) a maximum value for x in [–3, 10] ? (b) a minimum value for x in [–3, 10] ?

9. Prove that the function

$$f(x) = \begin{cases} x+1, & -1 \le x \le 0, \\ -x, & 0 < x \le 1 \end{cases}$$

is discontinuous at x = 0 and still has the maximum and minimum value on [–1, 1].

10. Show that there is a number x between $\pi/2$ and π such that tanx = –x.

11. Show that the given equation has atleast one solution on the indicated interval.

(i) $\sqrt[3]{x} = x^2 + 2x - 1$ on [0, 1]

(ii) $\dfrac{1}{x+1} = x^2 - x - 1$ on [1, 2]

(iii) $\sqrt[3]{x-8} + 9x^{2/3} = 29$ on [0, 8]

(iv) $\cos x = x^2 - 1$ on $[0, \pi]$

12. Prove that the equation tan x = x has infinite number of real roots.

13. Let $\phi(x) = \dfrac{x^{2n+2} - \cos x}{x^{2n} + 1}$, show that $\phi(0)$ and $\phi(2)$ differ in sign but $\phi(x)$ does not vanish in [0, 2].

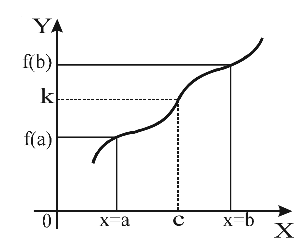
14. (i) Prove that the only polynomial function (with real coefficient) p such that, for all x > 0, $p(x)/x^2 \in [2, 3]$ are those of the form $p(x) = Ax^2$ for some constant $A \in [2, 3]$.

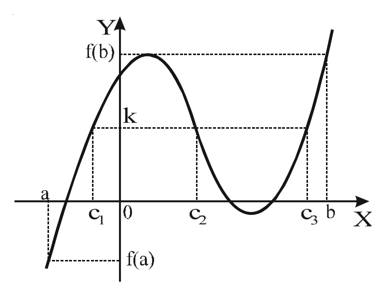
(ii) Give an example of a continuous function f from $(0, \infty)$ to R which is not a polynomial but such that, for all x > 0, $2 \le \dfrac{f(x)}{x^2} \le 3$.

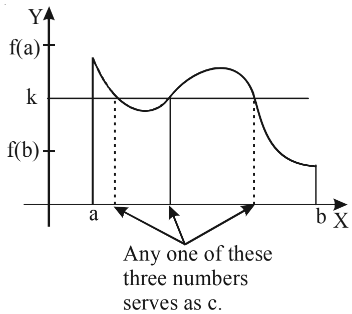
2.6 INTERMEDIATE VALUE THEOREM (I.V.T.)

Theorem If f is continuous on the closed interval [a, b] and k is any number between f(a) and f(b), then there is atleast one number c in [a, b] such that f(c) = k.

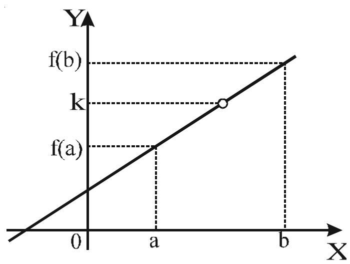
In other words, a continuous function defined on [a, b] takes on all values between f(a) and f(b). Pictorially, it asserts that a horizontal line of height k must meet the graph of f atleast once if k is between f(a) and f(b), as shown in the figure. That is, when you move a pencil along the graph of a continuous function from one height to another, the pencil passes through all intermediate heights. This is a way of saying that the graph has no gaps or jumps, suggesting that the idea of being able to trace such a graph without lifting the pen from the paper is accurate. The following graphs of functions continuous in a closed interval illustrate the theorem.







Note that continuity through the interval [a, b] is essential for the validity of this theorem. This is illustrated by the following figure



Even though the theorem guarantees the existence of c, it does not tell how to find it. Such theorems are called existence theorems. To find c, we must solve the equation, namely $f(c) = k$.

Proof Let us define a new function $g(x) = f(x) - k$, where k is a constant number lying between f(a) and f(b). Since f is a continuous function on [a, b], so is the function g.

Without loss of generality let us assume that $f(a) < k < f(b)$. Now $g(a) = f(a) - k < 0$ and $g(b) = f(b) - k > 0$. The function g, obviously, attains values of opposite signs at the end points of the interval [a, b]. Therefore, by Bolzano theorem , there is a point c lying inside (a, b) such that $g(c) = 0$ or $f(c) - k = 0$, that is $f(c) = k$. This is what we wished to prove.

Since k is any value between f(a) and f(b), it follows that f takes all values between f(a) and f(b) atleast once. In other words, a continuous function cannot pass from one value to another without assuming once (atleast) every intermediate value.

***Alternative proof*:** For definiteness, let $f(a) < 0$ and $f(b) > 0$. Divide the interval [a, b] into n equal parts and consider the values of the function with their proper sign at the points of division, say $x_1, x_2, \ldots, x_{n-1}$. Let x_{p+1} be the first of these points for which f(x) is positive. Then at $x = x_p$, the function is either negative or zero. If it is zero, the theorem is at once established. If not, we rename the interval $[x_p, x_{p+1}]$ as $[a_1, b_1]$.

Divide it into n equal parts and consider the values of f(x) at the points of division again. Suppose $[a_2, b_2]$ is the first of the new sub-intervals such that $f(b_2) > 0$ and $f(a_2) \le 0$.

We take $f(a_2) \neq 0$, for otherwise the theorem is established. Continuing this process indefinitely, we shall get a sequence of intervals

$[a_1, b_1], [a_2, b_2], [a_3, b_3], \ldots$

such that $[a_1, b_1) \supset [a_2, b_2] \supset [a_3, b_3] \supset \ldots$

Also $b_k - a_k = (b - a) / n^k \to 0$ as $k \to \infty$.

This sequence of intervals will therefore define a limiting point x_0. We will now show that $f(x_0) = 0$. Since f is continuous at $x = x_0$, we have $|f(x) - f(x_0)| < \in$ for all x in the interval $(x_0 - \delta, x_0 + \delta)$. Now we can choose m so large that the interval $[a_m, b_m]$ lies within $(x_0 - \delta, x_0 + \delta)$, such that $f(a_m) < 0$ and $f(b_m) > 0$. Also by the sign preserving property if $f(x_0) \neq 0$, f(x) must have the same sign as $f(x_0)$ in the interval $(x_0 - \delta, x_0 + \delta)$. Hence $f(a_m)$ and $f(b_m)$ cannot differ in sign unless $f(x_0)$ is zero, thus establishing the theorem.

As a simple example of this theorem, consider a person's height. Suppose that a girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

Example 1. Show that the function $f(x) = x - 1 - \cos \pi x, x \in [1, 2]$ takes on the value 1/2.

Solution The function $f(x) = x - 1 - \cos \pi x, x \in [1, 2]$ is continuous everywhere in the given interval.

Also, we have $f(1) = -1$ and $f(2) = 1$.

Since $-1 < 1/2 < 1$, hence by I.V.T. there exists atleast one point $x = c$ in [1, 2] such that $f(c) = 1/2$.

Example 2. Does the function $f(x) = x^3/4 - \sin \pi x + 3$ take on the value $2\frac{1}{3}$ within the interval [–2, 2]?

Solution The function $f(x) = x^3/4 - \sin \pi x + 3$ is continuous within the interval [–2, 2]. Furthermore, at the end points of this interval it attains the values $f(-2) = 1; f(2) = 5$.

Since $1 < 2\frac{1}{3} < 5$, then, by I.V.T. within the interval [–2, 2] there exists at least one point x such that $f(x) = 2\frac{1}{3}$.

Example 3. Show that the function $f(x) = (x-a)^2(x-b)^2 + x$ takes the value $\frac{a+b}{2}$ for some value of $x \in [a, b]$.

Solution f is continuous in [a, b] and f(a) = a, f(b) = b. We know that the average value of a and b

i.e. $\frac{a+b}{2} \in [a, b]$.

By Intermediate Value Theorem, there exists atleast one $c \in (a, b)$ such that $f(c) = \frac{a+b}{2}$.

Example 4. Use Intermediate Value Theorem to show that the equation $2x^3 + x^2 - x + 1 = 5$ has a solution in the interval [1, 2].

Solution Let $P(x) = 2x^3 + x^2 - x + 1$. Then

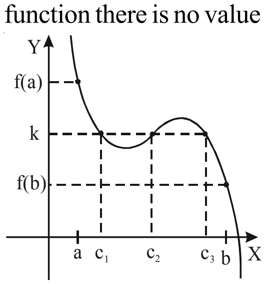
$P(1) = 2 \cdot 1^3 + 1^2 - 1 + 1 = 3$

$P(2) = 2 \cdot 2^3 + 2^2 - 2 + 1 = 19$

Since P is continuous and 5 is between P(1) = 3 and P(2) = 19, we may apply the Intermediate Value Theorem to P in the case a = 1, b = 2, and k = 5. Thus there is atleast one number c between 1 and 2 such that P(c) = 6. This completes the answer.

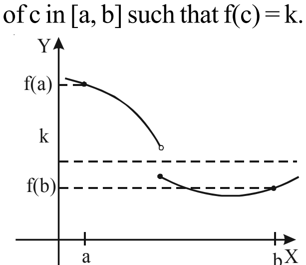
To get a more accurate estimate for a number c such that P(c) = 5, we can find a shorter interval for which the Intermediate Value Theorem can be applied. For instance, $P(1.2) \approx 4.7$ and $P(1.3) \approx 5.8$. By the intermediate value theorem, there is a number c in [1.2, 1.3] such that P(c) = 5.

Note: The Intermediate Value Theorem guarantees the existence of atleast one number c in the closed interval [a, b]. There may, of course, be more than one number c such that f(c) = k, as shown in Figure 1. A function that is not continuous does not necessarily possess the intermediate value property. For example, the graph of the function shown in Figure 2 jumps over the horizontal line given by y = k and for this function there is no value of c in [a, b] such that f(c) = k.



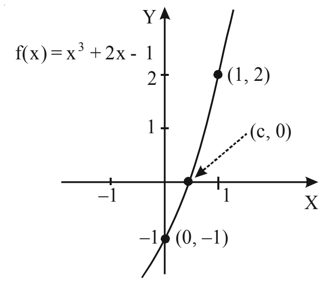
(Fig. 1)

f is continuous on [a, b].
(For k, there exist 3 c's.)



(Fig. 2)

f is not continuous on [a, b].
(For k, there are no c's.)



(Fig. 3)

f is continuous on [0, 1] with f(0) < 0 and f(1) > 0.

The converse of I.V.T. is however not true. Consider, for example, the function f defined as follows :

$f(x) = \sin(1/x), x \neq 0$ and $f(0) = 0$.

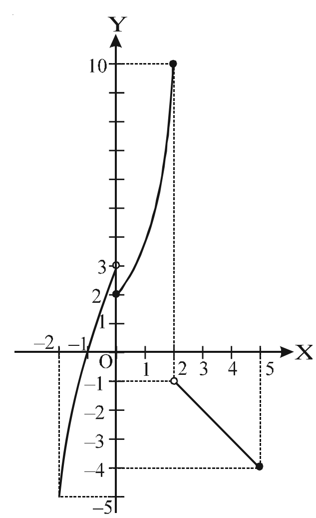
In the interval $[-2/\pi, 2/\pi]$, this function takes all values between $f(-2/\pi)$ and $f(2/\pi)$, that is, between –1 and 1 and infinite number of times as x varies from $-2/\pi$ to $2/\pi$ but the function is not continuous in this interval, being discontinuous at x = 0.

Example 5. Let

$$f(x) = \begin{cases} 3 - x^2 + 2x & -2 \le x < 0 \\ x^3 + 2 & 0 \le x \le 2 \\ 1 - x & 2 < x \le 5 \end{cases}$$

Show that f(x) attains all intermediate values between f(–2) and f(5) even if f is discontinuous in the interval.

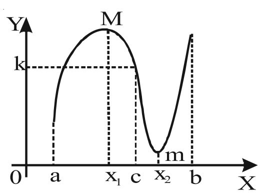
Solution This is self evident from the graph of the function shown below. Howevever, this could not be concluded from I.V.T. since f(x) is discontinuous at x = 0 and x = 2.



Corollary. The I.V.T can also be reformulated in the following way : a function continuous on a closed interval [a, b] assumes all the intermediate values lying between its least and greatest values on [a, b] (which exist by virtue of Extreme Value Theorem).

In other words, let f be continuous on [a, b] an let $k \in [m, M]$ where m = absolute minimum value of f and M = absolute maximum value of f on [a, b]. Then there exists $c \in [a, b]$ such that $f(c) = k$.

Proof By extreme value theorem, there exist $x_1, x_2 \in [a, b]$ such that $m = f(x_1)$ and $M = f(x_2)$. If $x_1 = x_2$, then f is constant on [a, b] and the result follows. Let $x_1 < x_2$. Since, f is continuous on $[x_1, x_2] \subset [a, b]$, by intermediate value theorem there exists $c \in [x_1, x_2] \subset [a, b]$ such that $f(c) = k$.



Similarly the result can be proved when $x_1 > x_2$.

Note: A continuous function whose domain is closed must have a range in a closed interval but it is not necessary that if the domain is open then range is also open (range can be closed).

Theorem Let f be a continuous strictly increasing function on a closed interval [a, b] and let $\alpha = f(a)$, $\beta = f(b)$. Then (i) the range of f for the closed interval [a, b] is the closed interval $[\alpha, \beta]$, (ii) there exists a function $x = g(y)$, the inverse of f, which is one-valued, strictly increasing and continuous on $[\alpha, \beta]$.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $y = \sin x$ is continuous and strictly increasing; it has a continuous inverse which is, as we know, designated as

$$x = \sin^{-1} y \ (-1 \le y \le 1).$$

A strictly decreasing function f(x), continuous on [a, b] has an inverse, which is a strictly decreasing continuous function on $[\beta, \alpha]$ where $\alpha = f(a)$, $\beta = f(b)$.

Theorem Let a strictly increasing function $y = f(x)$ map the closed interval [a, b] onto the closed interval $[\alpha, \beta]$, i.e. $f([a, b]) = [\alpha, \beta]$. Then f is continuous on [a, b].

Proof Let us be given an arbitrary point x_0 belonging for the time being to the open interval (a, b). By virtue of the fact that f is strictly increasing, the corresponding point $y_0 = f(x_0)$ will belong to the interval (α, β) $(\alpha < y_0 < \beta)$.

Let us take $\varepsilon > 0$ so small that $\alpha < y_0 - \varepsilon < y_0 < y_0 + \varepsilon < \beta$. By the hypothesis, there exist points $x_1, x_2 \in (a, b)$, $(x_1 < x_0 < x_2)$ such that $y_0 - \varepsilon = f(x_1)$, $y_0 + \varepsilon = f(x_2)$. The interval (x_1, x_2) can be regarded as a neighbourhood of the point x_0 $(x_0 \in (x_1, x_2))$.

Since the function f increases, for $x \in (x_1, x_2)$ we shall have $y_0 - \varepsilon < f(x) < y_0 + \varepsilon$ or $|f(x) - y_0| < \varepsilon$, i.e. $|f(x) - f(x_0)| < \varepsilon$, and we have proved the continuity of the function f at the point x_0.

If $x_0 = a$ or $x_0 = b$, then we prove the one-sided continuity of the function f in a similar way.

Let f(x) be a function defined and bounded in the interval [a, b], then, if M and m are the bounds of f(x), the number M – m is called the **span** or **oscillation** of the function f(x) in the interval.

Theorem A continuous function attains its bounds. If f(x) is continuous in [a, b] and M and m are its maximum and minimum values, there are atleast two points x_1 and x_2 in (a, b) such that

$$f(x_1) = M, f(x_2) = m.$$

Proof Suppose that M is not attained; then $M - f(x)$ does not vanish at any point of [a, b]. Hence $\dfrac{1}{M - f(x)}$ is a continuous function, and therefore bounded. If $G > 0$ be its upper bound, we have

$$\frac{1}{M - f(x)} \le G,$$

So that $$M - f(x) \ge \frac{1}{G},$$

that is, $$f(x) \le M - \frac{1}{G}$$

but this contradicts the fact that M is the maximum value of f(x) in (a, b). Hence M must be attained. Similarly it may be proved that m is attained.

Consider the following examples:

(i) If $f(x) = 1/x$ except when $x = 0$ and $f(0) = 0$, then f(x) has neither an upper nor a lower bound in any interval which includes $x = 0$ in its interior, such as the interval (–1, 1).

(ii) If $f(x) = 1/x^2$ except when $x = 0$, and $f(0) = 0$, then f(x) has the lower bound 0, but no upper bound, in the interval (–1, 1).

(iii) Let $f(x) = \sin(1/x)$ except when $x = 0$ and $f(0) = 0$, then f(x) is discontinuous for $x = 0$. In any interval $(-\delta, \delta)$ the lower bound is –1 and the upper bound 1 and each of these values is assumed by f(x) an infinity of times.

(iv) Let $f(x) = x - [x]$. This function is discontinous for all integral values of x. In the interval (0, 1) its lower bound in 0 and its upper bound 1. It is equal to 0. when $x = 0$ or $x = 1$, but it is never equal to 1. Thus f(x) never assumes a value equal to its upper bound.

(v) Let $f(x) = 0$ when x is irrational, and $f(x) = q$ when x is a rational fraction p/q. Then f(x) has the lower bound 0, but no upper bound, in any interval (a, b).

But if $f(x) = (-1)^p q$ when $x = p/q$ then f(x) has neither an upper nor a lower bound in any interval.

Corollary. If f(x) is continuous in the interval [a, b], then (a, b) can be subdivided into a finite number of sub-intervals in each of which the span or oscillation of f(x) is less than any given ε.

Existence of Solutions of Equations

The Intermediate Value Theorem can be often used to locate the zeros of a function (i.e.solutions of equations written in the form f(x) = 0) that is continuous on a closed interval. Specifically, if f is continuous on [a, b] and f(a) and f(b) differ in sign, then the Intermediate Value Theorem guarantees the existence of atleast one zero of f in the open interval (a, b).

Let $f(x) = x^2 - 2 = 0$

$f(1) = -1 < 0$, whereas $f(2) = 2 > 0$

We note that the function f is continuous on [1, 2] and that k = 0 is an intermediate value of f on the interval [1, 2]. Therefore, it follows from I.V.T. that $f(c) = c^2 - 2 = 0$ for some number c in (1, 2).

i.e. $c^2 = 2$

This number c is the desired square root of 2. Thus it is the intermediate value property of continuous functions that guarantees the existence of the number $\sqrt{2}$.

Consider another example. Prove that the equation $x - \cos x = 0$ possesses a root in the interval $(0, \pi)$.

The function $f(x) = x - \cos x$ is continuous on the closed interval $[0, \pi]$ and assumes $f(0) = -1$, and $f(\pi) = \pi + 1$ values having opposite signs at its end points. Since $-1 < 0 < \pi + 1$, by I.V.T. f(x) assumes the value 0 at some x between 0 and π. Thus, the equation $x - \cos x = 0$ possesses a root in the interval $(0, \pi)$.

Example 6. Show that the equation $x^5 - 2x^2 + x + 11 = 0$ has atleast one real root.

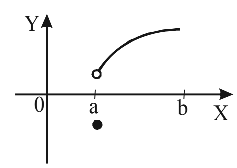
Solution For x large and positive the polynomial $P(x) = x^5 - 2x^2 + x + 11$ is positive (since $\lim_{x\to\infty} P(x) = \infty$).

Thus, there is a number b such that $P(b) > 0$. Similarly, for x negative and of large absolute value P(x) is negative (since $\lim_{x} P(x) = -\infty$). Let us now select a number a such that $P(a) < 0$.

The number 0 is between P(a) and P(b). Since P is continuous on the interval [a, b]. There is a number c in [a, b] such that $P(c) = 0$. This number c is a real solution to the equation $x^5 - 2x^2 + x + 11 = 0$.

Note that the argument in this example applies to any polynomial of odd degree.

Note: In both Bolzano's theorem and the intermediate value theorem, it is assumed that f is continuous at each point of [a, b], including the endpoints a and b. To understand why continuity at both endpoints is necessary, we refer to the curve in the figure below.



Here f is continuous everywhere in (a, b], excluding the end point a. Although f(a) is negative and f(b) is positive, there is no x in (a, b) for which $f(x) = 0$.

Example 7. If n is a positive integer and if $a > 0$, then prove that there is exactly one positive b such that $b^n = a$.

Solution Choose $c > 1$ such that $0 < a < c$, and consider the function f defined on the interval [0, c] by the equation $f(x) = x^n$. This function is continuous on [0, c], and at the end points we have $f(0) = 0$, $f(c) = c^n$.

Since $0 < a < c < c^n$, the given number a lies between the function values f(0) and f(c). Therefore, by the Intermediate Value Theorem, we have $f(x) = a$ for some x in (0, c) say for x = b. This proves the existence of atleast one positive b such that $b^n = a$. There cannot be more than one such b because f is strictly increasing on [0, c]. This completes the proof.

Example 8. Prove that the equation

$$\sqrt{x-5} = \frac{1}{x+3} \text{ has atleast one real root.}$$

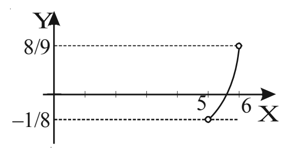
Solution Let $f(x) = \sqrt{x-5} - \frac{1}{x+3}$

The function is continuous on [5, 6].

$$\text{Also } f(5) = 0 - \frac{1}{5+3} = -\frac{1}{8} < 0$$

$$f(6) = 1 - \frac{1}{9} = \frac{8}{9} > 0$$

Hence by intermediate value theorem there exists atleast one value of $c \in (5, 6)$ for which $f(c) = 0$



$$\therefore \quad \sqrt{c-5} - \frac{1}{c+3} = 0$$

Thus, c is root of the equation $\sqrt{x-5} = \frac{1}{x+3}$

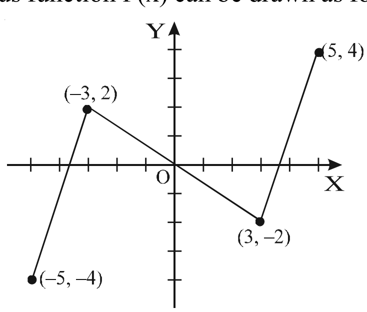
where $c \in (5, 6)$.

Example 9. Let $f : R \to R$ be a continuous onto function satisfying $f(x) + f(-x) = 0, \forall\ x \in R$. If $f(-3) = 2$ and $f(5) = 4$ in $[-5, 5]$, then prove that the equation $f(x) = 0$ has atleast three real roots.

Solution $f(x) + f(-x) = 0 \Rightarrow f(x)$ is an odd function.

Since points (– 3, 2) and (5, 4) lie on the curve, points (3, – 2) and (– 5, – 4) will also lie on the curve.

For minimum number of roots, the graph of a continuous function f (x) can be drawn as follows.



From the above graph of f(x), it is clear that equation f(x) = 0 has atleast three real roots.

Example 10. Using Intermediate Value Theorem, prove that there exists a number x such that

$$x^{200} + \frac{1}{1+\sin^2 x} = 200.$$

Solution Let $f(x) = x^{200} + (1 + \sin^2 x)^{-1}$.

f is continuous and $f(0) = 1 < 200$ and $f(2) > 2^{200}$, which is much greater than 200. Hence, from the Intermediate Value Theorem there exists a number c in (0, 2) such that $f(c) = 200$.

Example 11. Let $f : R \to R$ satisfy

$$f(x) - f(y) = e^{x-y} - 1 \ \forall\ x, y \in R.$$

Prove that f is a continuous function. Also prove that the function f(x) has atleast one zero if $f(0) < 1$.

Solution $\lim_{h\to 0} f(x+h) - f(x) = \lim_{h\to 0} (e^{x+h-x} - 1) = 0$

Hence f is continuous everywhere.

Putting y = 0, we have $f(x) = f(0) + e^x - 1$

Also $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = f(0) - 1 < 0$

Since f(x) is positive for large positive x and negative of large negative x, by Intermediate Value Theorem f(x) = 0 has atleast one root.

Example 12. If f(x) be a continuous function in $[0, 2\pi]$ and $f(0) = f(2\pi)$ then prove that there exists point $c \in (0, \pi)$ such that $f(c) = f(c + \pi)$.

Solution Let $g(x) = f(x) - f(x + \pi)$ ....(1)

at $x = \pi$; $\quad g(\pi) = f(\pi) - f(2\pi)$ ....(2)

at $x = 0$, $\quad g(0) = f(0) - f(\pi)$ ...(3)

Adding (2) and (3), $g(0) + g(\pi) = f(0) - f(2\pi)$

$\Rightarrow$ $g(0) + g(\pi) = 0$ [given $f(0) = f(2\pi)$]

$\Rightarrow$ $g(0) = -g(\pi)$

$\Rightarrow$ g(0) and $g(\pi)$ are opposite in sign.

$\Rightarrow$ There exists a point c between 0 and π such that $g(c) = 0$.

From (1) putting $x = c$, $g(c) = f(c) - f(c + \pi) = 0$

Hence, $\quad f(c) = f(c + \pi)$.

Example 13. Suppose that f is a continuous function of [0, 1] and that f(0) = f(1). Let n be a positive integer. Prove that there is some number $x \in [0, 1]$ such that $f(x) = f(x + 1/n)$. [Universal Chord Theorem]

Solution Consider $g(x) = f(x) - f(x + 1/n)$, which is clearly continuous. If g is never zero in [0, 1] then g must be either strictly positive or strictly negative. But then

$$0 = f(0) - f(1) = \left(f(0) - f\left(\frac{1}{n}\right)\right) + \left(f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right)\right)$$

$$+ \left(f\left(\frac{2}{n}\right) - f\left(\frac{3}{n}\right)\right) + \ldots + \left(f\left(\frac{n-1}{n}\right) - f\left(\frac{n}{n}\right)\right).$$

The sum of each parenthesis on the right is strictly positive or strictly negative and hence never 0, a contradiction.

Example 14. A function f is continuous in the interval [0,1] and assumes only rational values in the entire interval. If $f(\frac{1}{2}) = \frac{1}{2}$, prove that $f(x) = \frac{1}{2}$ everywhere.

Solution Suppose, if possible, that there exists a point $c \in [0,1]$ such that $f(c) \neq \frac{1}{2}$. It is obvious that $c \neq \frac{1}{2}$. By the law of trichotomy, either $c < \frac{1}{2}$ or $\frac{1}{2} < c$. Without loss of generality, let us assume that $c < \frac{1}{2}$. The function f is continuous in $[c, \frac{1}{2}]$. and therefore by I.V.T. it must every value lying between f(c) and $f\left(\frac{1}{2}\right)$. But this is not possible, because f(c) and $f\left(\frac{1}{2}\right)$ are two distinct rational numbers between which there lie infinitely many irrational numbers and f(x) does not take any irrational value. The contradiction shows that there does not exist any $c \in [0, 1]$ such that f(c) is different from $\frac{1}{2}$. Hence $f(x) = \frac{1}{2}$ everywhere.

Example 15. Let f and g be continuous functions on an interval I, let $f(x) \neq 0$ for any $x \in I$ and let $(f(x))^2 = (g(x))^2$ for all $x \in I$. Prove that either $f(x) = g(x)$ for all $x \in I$ or $f(x) = -g(x)$ for all $x \in I$.

Solution Suppose, if possible, that there exist $x_1 \in I$ and $x_2 \in I$, such that $f(x_1) = g(x_1)$ and $f(x_2) = -g(x_2)$. Since f is continuous on I and $f(x) \neq 0$ anywhere on I, therefore $f(x_1)$ and $f(x_2)$ must be of the same sign. Consequently, $g(x_1)$ and $g(x_2)$ must be of opposite signs. Now g is continuous on I and $g(x_1)$, $g(x_2)$ are of opposite signs. Therefore, by Intermediate Value Theorem, there exists x_0 lying between x_1 and x_2, such that $g(x_0) = 0$. Combining this with $(f(x))^2 = (g(x))^2$ for all $x \in I$, we have $f(x_0) = 0$, wheich is not possible.

Therefore either $f(x) = g(x)$ for all $x \in I$ or $f(x) = -g(x)$ for all $x \in I$.

Example 16. Let f be a continuous function defined over the real numbers that satisfies the Cauchy functional equation $f(x + y) = f(x) + f(y)$, $\forall\, x, y \in R$, Then prove that f is linear, that is, there is a constant c such that $f(x) = cx$.

Solution We first prove the assertion for positive integers n using induction. We then extend our result to negative integers. Then we extend the result to reciprocals of integers and after that to rational numbers. Finally we extend the result to all real numbers.

We prove by induction that for integer $n \geq 0$, $f(nx) = nf(x)$. Using the functional equation,

$$f(0 \cdot x) = f(0 \cdot x + 0 \cdot x) = f(0 \cdot x) + f(0 \cdot x)$$

$$\Rightarrow \quad f(0 \cdot x) = 0\, f(x),$$

and the assertion follows for $n = 0$. Assume $n \geq 1$ is an integer and that $f((n-1)x) = (n-1)f(x)$.

Then

$$f(nx) = f((n-1)x + x) = f((n-1)x) + f(x)$$
$$= (n-1)\, f(x) + f(x) = nf(x),$$

proving the assertion for all strictly positive integers.

Let $m < 0$ be an integer. Then $-m > 0$ is a strictly positive integer, for which the result proved above holds, and thus, $f(-mx) = -mf(x)$. Now.

$$0 = f(0) \Rightarrow 0 = f(mx + (-mx)) = f(mx) + f(-mx)$$

$$\Rightarrow \quad f(mx) = -f(-mx) = -(mf(x)) = mf(x),$$

and the assertion follows for negative integers.

We have thus proved the theorem for all integers.

Assume now that $x = \frac{a}{b}$, with $a \in I$ and $b \in I - \{0\}$.

Then $f(a) = f(a \cdot 1) = af(1)$ and

$f(a) = f\left(b\frac{a}{b}\right) = bf\left(\frac{a}{b}\right)$ by the result we proved for integers and hence

$$af(1) = bf\left(\frac{a}{b}\right) \Rightarrow f\left(\frac{a}{b}\right) = f(1)\left(\frac{a}{b}\right).$$

We have established that for all rational numbers $x \in Q$, $f(x) = xf(1)$.

We have not used the fact that the function is continuous so far. Since the rational are dense in the reals the extension of the result to all real numbers now follows.

Example 17. f is a continuous real-valued function such that $f(x + y) = f(x) f(y)$ for all real x, y. If $f(2) = 5$, find f(5).

Solution More generally, since $f(nx) = f(x)^n$ for all integers n, $f(1) = c = f(1/n)^n$ for some constant c and all integers n. Thus $f(k/n) = f(1/n)^k = f(1)^{k/n} = c^{k/n}$ for all rational numbers k/n. By continuity, it follows that $f(x) = c^x$ for all real numbers x. Since $f(2) = 5$, $c = \sqrt{5}$, so $f(5) = 25\sqrt{5}$.

Example 18. Let f(x) be a polynomial with real coefficients for which the equation $f(x) = x$ has no real solution. Prove that the equation $f(f(x)) = x$ has no real solution either.

Solution Let $g(x) = f(x) - x$. Then, g(x) is a polynomial that never vanishes. We argue that it must always have the same sign. Suppose if possible that $g(a) < 0 < g(b)$ for some reals a and b. Since g(x), being a polynomial, is continuous, the Intermediate Value Theorem applies and there must be a number c between a and b for which $g(c) = 0$, yielding a contradiction.

Thus, either $g(x) > 0$ for all x or else $g(x) < 0$ for all x. Then,

$$f(f(x)) - x = f(f(x)) - f(x) + f(x) - x$$
$$= g(f(x)) + g(x)$$

for all real x. Since g never changes sign, both g(x) and g(f(x)) have the same sign (either positive or negative) and so their sum cannot vanish. Hence $f(f(x)) \neq x$ for any real x.

Concept Problems F

1. Use the Intermediate Value Theorem to show that there is a number c in (1, 2] such that $4 - c = 2^c$.
2. Verify that the Intermediate Value Theorem applies in the indicated interval and find the value of c guaranteed by the theorem.
   (i) $f(x) = x^2 + x - 1$, [0, 5], $f(c) = 11$
   (ii) $f(x) = x^3 - x^2 + x - 2$, [0, 3], $f(c) = 4$
   (iii) $f(x) = \dfrac{x^2 + x}{x - 1}$, $\left[\dfrac{5}{2}, 4\right]$, $f(c) = 6$
3. Use the Intermediate Value Theorem to show that the equation has atleast one root :
   (i) $2x^3 + x^2 - x = 4$ (ii) $x^5 - 2x^2 + x + 11 = 0$
4. Prove that the equation $2x^3 + 5x^2 - 5x - 3 = 0$ has a root between $-\infty$ and –1, another between –1 and 0 and a third between 1 and 2.
5. Show that the equation $x^3 - 3x + 1 = 0$ has a real root in the interval (1, 2). Approximate this root.
6. Show that the equation $x^3 - 3x^2 + 1 = 0$ has three distinct roots by calculating the values of the function at x = –3, –2, –1, 0, 1, 2 and 3 and then applying the Intermediate Value Theorem of continuous functions on appropriate closed intervals.
7. Prove that $x^3 + x^2 - 3x - 3 = 0$ has a root between 1 and 2, between 1.5 and 1.75, between 1.625 and 1.75, etc. Show that if we continue this procedure, we can approximate the root of the equation as closely as we want.
8. If $f(x) = x^3 - x^2 + x$, show that there is a number c such that $f(c) = 10$.
9. Use the Intermediate Value Theorem to show that there is a root of the given equation in the specific interval :
   (i) $\sqrt[3]{x} = -1 - x$, (0, 1) (ii) $\ln x = e^{-x}$, (1, 2)
10. Is there a number that is exactly 1 more than its cube?
11. Apply the Intermediate Value Theorem to prove that every real number has a cube root.
12. Apply the Intermediate Value Theorem to prove that the equation $x^5 + x = 1$ has a solution.
13. Apply the Intermediate Value Theorem to prove that the equation $x^3 - 4x^2 + 1 = 0$ has three solutions.
14. Let f be continuous on [a, b] and let f(x) be always rational. What can we say about f ?
15. Suppose that f and g are two functions both continuous on the interval [a, b], and such that $f(x) = g(b) = p$ and $f(b) = g(a) = q$ where $p \neq q$. Apply Intermediate Value Theorem to the function $h(x) = f(x) - g(x)$ to show that $f(c) = g(c)$ at some point c of (a, b).
16. Suppose that today you leave your home at 1 P.M. and drive to school, arriving at 2 P.M.

Tomorrow you leave your school at 1 P.M. and retrace the same route, arriving home at 2 P.M. Show that at some instant between 1 and 2 P.M. you are at precisely the same point on the road both days.

17. Let $f : [1, e] \to [0,1]$ be continuous then prove that $f(x) = \ln x$ has atleast one solution in $[1, e]$.

18. Let $f : (1, 10) \to [2,11]$ be a continuous function, then prove that it cannot be an invertible function.

19. Let f be continuous on [a, b] and suppose that $f(x) = 0$ for every rational x in [a, b]. Prove that $f(x) = 0$ for all x in [a, b].

20. Let f, g be continuous functions from R to R. Suppose that $f(x) = g(x)$ for all $x \in Q$. Prove that $f(x) = g(x)$ for all $x \in R$, i.e. $f = g$.

Practice Problems F

21. Show that the equation

$$x + \sin x = \frac{1}{\sqrt{x+3}}$$

has atleast one solution on the interval $[0, \pi]$.

22. Show that the equation $x^5 + 3x^4 + x - 2 = 0$ has atleast one root in the interval [0, 1].

23. Show that the equation $x^5 - 2x^3 + x^2 - 3x + 1 = 0$ has atleast one root in the interval [1, 2].

24. Find roughly the situations of the roots of $2x^3 - 3x^2 - 36x + 10 = 0$.

25. Apply the Intermediate Value Theorem to show that every positive number a has a square root. That is, given $a > 0$, prove that there exists a number r such that $r^2 = a$.

26. Suppose that $a < b < c$. If the function f is continuous on the closed interval [a, b] and on the closed interval [b, c], does it follow that f is continuous on [a, c] ? If f is continuous on the closed interval [n, n + 1] for every integer n, does it follow that f is continuous on the entire real line?

27. Let f be a continuous function on R and periodic with fundamental period 1 i.e. $f(x + 1) = f(x)$, then prove that there will be a real number x_0, such that $f(x_0 + \pi) = f(x_0)$.

28. Use the Intermediate Value Theorem to show that there is a square with a diagonal length that is between r and 2r and an area that is half the area of a circle of radius r.

29. Let $f : [0, 1] \to [0, 1]$ and $g : [0, 1] \to [0, 1]$ be continuous fractions. Given that $f(0) < g(0)$ and $(f(1))^3 > g(1)$ prove that there is a number c in (0, 1) such that $(f(c))^3 = g(c)$

30. Prove that $\sqrt{x+3} = e^x$ has atleast one real root.

31. Let f be a polynomial of degree n, say $f(x) = \sum_{k=0}^{n} c_k x^k$, such that the first and last coefficients c_0 and c_n have opposite signs. Prove that $f(x) = 0$ for atleast one positive x.

32. Given a real valued function f which is continuous on the closed interval [a, b]. Assume that $f(a) \le a$ and that $f(b) \ge b$. Prove that f has a fixed point in [a, b].

33. Prove that if f is continuous and has no zeros in [a, b], then either $f(x) > 0$ for all x in [a, b] or $f(x) < 0$ for all x in [a, b].

34. A rational function can have infinitely many x-values at which it is discontinuous. True or false.

35. Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interval [a,b]. If $f_1(a) < f_2(a)$ and $f_2(b) > f_2(b)$, prove that there exists c between a and b such that $f_1(c) = f_2(c)$.

36. Show that the equation $x^4 + 5x^3 + 5x - 1 = 0$ has atleast two solutions in the interval [–6, 2].

37. If f(x) is a continuous function in [2, 3] which takes only irrational values for all $x \in [2, 3]$ and $f(2.5) = \sqrt{5}$ then find f(2.8).

38. Find an interval in which the given equation has atleast one solution. Note that the interval is not unique.

(i) $\ln x = (x - 2)^2$ (ii) $e^{-x} = x^3$

(iii) $\cos x - \sin x = x$ (iv) $\tan x = 2x^2 - 1$

39. Let f be a continuous function from R to R. Suppose that (i) $f(1) = 1$ and that (ii) $f(x + y) = f(x) + f(y)$ for all real x and y.

(a) Prove that $f(x) = x$ for every positive rational x.

(b) Prove that $f(0) = 0$ and that $f(-x) = -f(x)$ for all real x.

(c) Prove that $f(x) = x$ for all real x.

40. Let f be a continuous function such that f(x) f(f(x)) = 1 and f(2006) = 2005, then consider the following assertion : $f(1000) = \frac{1}{1000}$. If the reason is : as $f(f(x)) = \frac{1}{f(x)}$ we have $f(t) = \frac{1}{t}$

$\Rightarrow f(1000) = \frac{1}{1000}$

Prove that the assersion is correct but the reason is false.

41. Find all functions $f : R \to R$, continuous at x = 1 such that $\forall x \in R, f(x) = -f(x^2)$.

42. Let f and g be continuous functions defined for all x. Assume that f(x) = g(x) for all rational x. Deduce that f(x) = g(x) for all real numbers x.

43. Let us determine all continuous functions f such that f(x + y) = f(x)f(y) for all real numbers x and y and such that the values of f are always positive.

(a) Let b be a fixed positive number. Let $f(x) = b^x$. Check that f(x + y) = f(x)f(y). [Part(b) will show that there are no other functions that satisfy the stated conditions]

(b) Assume that f is a continuous function such that f(x) > 0 for all x and f(x + y) = f(x) f(y) for all x and y. Let f(1) = c.

(i) Show that $f(n) = c^n$ for any positive integer n.

(ii) Show that f(0) = 1.

(iii) Show that $f(n) = c^n$ for any negative integer n.

(iv) Show that $f(1/n) = \sqrt[n]{c}$ for any positive integer n.

(v) Show that $f(m/n) = (\sqrt[n]{c})^m$ for any integer m and positive integer n.

(vi) By (v), $f(x) = c^x$ for any rational number x. Assuming that f is continuous and that the exponential function c^x is continuous, deduce that $f(x) = c^x$ for all real numbers x.

44. Let f be a continuous function whose domain is the x axis and which has the property that f(x + y) = f(x) + f(y) for all numbers x and y. This question shows that f must be of the form f(x) = cx for some constant c.

(a) Let f(1) = c. Show that f(2) = 2c.

(b) Show that f(0) = 0.

(c) Show that f(–1) = –c.

(d) Show that for any positive integer n, f(n) = cn.

(e) Show that for any negative integer n, f(n) = cn.

(f) Show that $f(\frac{1}{2}) = c/2$.

(g) Show that for any nonzero integer n, f(1/n) = c/n.

(h) Show that for nay integer m and positive integer n, f(m/n) = c(m/n).

(i) Show that for any irrational number x, f(x) = cx. (This is where the continuity of f enters).

Target Exercises for JEE Advanced

Problem 1. Find the values of a and b if f is continuous at x = π/2, where

$$f(x) = \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}}, \qquad 0 < x < \frac{\pi}{2}$$

$$= a + 2, \qquad x = \frac{\pi}{2}$$

$$= (1 + |\cot x|)^{\frac{b|\tan x|}{a}}, \qquad \frac{\pi}{2} < x < \pi$$

Solution We have $f\left(\frac{\pi}{2}\right) = a + 2$

$$\lim_{h\to 0} f\left(\frac{\pi}{2} - h\right) = \lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{\tan 6(\pi/2 - h)}{\tan(5\pi/2 - 5h)}}$$

$$= \lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{\tan(3\pi - 6h)}{\tan(5\pi/2 - 5h)}} = \lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{-\tan 6h}{\cot 5\pi}}$$

$$= \lim_{h\to 0}\left(\frac{6}{5}\right)^{-\tan 6h \tan 5h} = \lim_{h\to 0}\left(\frac{6}{5}\right)^{0} = 1$$

$$\lim_{h\to 0} f\left(\frac{\pi}{2} + h\right) = \lim_{h\to 0}\left(1 + \left|\cot\left(\frac{\pi}{2} + h\right)\right|\right)^{\frac{b|\tan(\pi/2 + h)|}{a}}$$

$$= \lim_{h\to 0}(1 + \tan h)^{\frac{b\coth}{a}} = e^{b/a}.$$

Since f is continuous at x = π/2, we have

$$a + 2 = 1 = e^{b/a}$$

which gives $a = -1, b = 0$

Problem 2. Obtain a relation in a and b, if possible, so that the function

$$f(x)=\lim_{n\to\infty}\frac{x^n(a+\sin(x^n))+(b-\sin(x^n))}{(1+x^n)\sec(\tan^{-1}(x^n+x^{-n}))}$$

is continuous at x = 1.

Solution $f(x)=\lim_{n\to\infty}\frac{x^n(a+\sin(x^n))+(b-\sin(x^n))}{(1+x^n)\sec(\tan^{-1}(x^n+x^{-n}))}$

For continuity at x = 1

$\lim_{x\to1} f(x)$ must exist and equals $f(1)$

$$f(1)=\lim_{n\to\infty}\frac{1^n(a+\sin 1^n)+b-\sin(1^n)}{(1+1^n)\cdot\sec\left(\tan^{-1}(1^n+1^{-n})\right)}$$

$$=\frac{a+\sin 1+b-\sin 1}{\sec(\tan^{-1}2)}=\frac{a+b}{2\sqrt{5}}$$

Now for x > 1 in the immediate neighbourhood

$$f(x)=\lim_{n\to\infty}\frac{a+\sin(x^n)+\dfrac{b-\sin x^n}{x^n}}{\left(1+\dfrac{1}{x^n}\right)\sec(\tan^{-1}(x^n+x^{-n}))}$$

$$=\frac{a+(\text{some quantity between 1 and }-1)+0}{1\cdot\sec(\tan^{-1}\infty)}=0$$

Similarly for x < 0 in the immediate neighbourhood of 0

$$f(x)=\frac{b}{1\cdot\sec(\tan^{-1}\infty)}=0$$

Hence $f(x)=0$ for $x\neq1$

$\therefore\ \lim_{x\to1} f(x)=0=a+b.$

Problem 3. If $f(x)=\frac{A\cos x+Bx\sin x-5}{x^4}$, $(x\neq0)$ is continuous at x = 0, then find the value of A and B. Also find f(0).

Solution For continuity $\lim_{x\to0} f(x)=f(0)$

Now for $\lim_{x\to0}\frac{A\cos x+Bx\sin x-5}{x^4}$ to exist

as $x\to0$, Numerator $\to A-5$ and Denominator $\to0$. Hence $A-5=0 \Rightarrow A=5$

Hence $\lim_{x\to0}\frac{Bx\sin x-5(1-\cos x)}{x^4}$

$$=\lim_{x\to0}\frac{B\cdot\dfrac{\sin x}{x}-\dfrac{5}{1+\cos x}\dfrac{\sin^2 x}{x^2}}{x^2}$$

as $x\to0$, Numerator $\to B-\frac{5}{2}$ and

Denominator $\to0 \Rightarrow B=\frac{5}{2}$

$$\lim_{x\to0}\frac{x\sin x-2(1-\cos x)}{x^4}$$

$$=\frac{5}{2}\lim_{x\to0}\frac{2x\sin\frac{x}{2}\cos\frac{x}{2}-4\sin^2\frac{x}{2}}{x^4}$$

$$=\frac{5}{2}\lim_{x\to0}\frac{2\sin\frac{x}{2}}{x}\lim_{x\to0}\frac{x\cos\frac{x}{2}-2\cos\frac{x}{2}}{x^3}$$

Let $x=2\theta$

$$=\frac{5}{16}\lim_{\theta\to0}\frac{2\theta\cos\theta-2\sin\theta}{\theta^3}$$

$$=\frac{5}{16}\lim_{\theta\to0}2\cos\theta\frac{(\theta-\tan\theta)}{\theta^3}$$

$$=\frac{5}{8}\lim_{\theta\to0}\frac{\theta-\tan\theta}{\theta^3}=\frac{5}{8}\left(-\frac{1}{3}\right)=-\frac{5}{24}.$$

Since f is continuous $f(0)=-\frac{5}{24}$.

Problem 4. Let S denotes the sum of an infinite geometric progression whose first term is the value of the function $f(x)=\frac{\sin(x-(\pi/6))}{\sqrt{3}-2\cos x}$ at $x=\pi/6$, if f(x) is continuous at $x=\pi/6$ and whose common ratio is the limiting value of the function $g(x)=\frac{\sin(x)^{1/3}\ln(1+3x)}{(\tan^{-1}\sqrt{x})^2(e^{5x^{1/3}}-1)}$ as $x\to0$. Find the value of S.

Solution $a=f\left(\frac{\pi}{6}\right)=\lim_{x\to\pi/6}\frac{\sin(x-(\pi/6))}{\sqrt{3}-2\cos x}$

$$=\lim_{x\to\pi/6}\frac{\sin(x-(\pi/6))}{2(\cos(\pi/6)-\cos x)}$$

$$=\lim_{x\to\pi/6}\frac{2\sin((x/2)-(\pi/12))\cos((x/2)-(\pi/12))}{4\sin((\pi/12)+(x/2))\sin((\pi/12)+(x/2))}$$

$$=\frac{2}{2}=1.$$

Hence, a = 1

$$r=\lim_{x\to 0}\frac{\sin(x)^{1/3}.\ln(1+3x)}{\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\right)^2\left(e^{5.x^{1/3}}-1\right)}\frac{\left(5x^{1/3}\right)}{\left(5x^{1/3}\right)}$$

$$=\lim_{x\to 0}\frac{3\ln(1+3x)^{1/3x}}{5}=\frac{3}{5}.$$

$$\therefore\ \text{Sum } S=\frac{a}{1-r}=\frac{1}{1-\frac{3}{5}}=\frac{5}{2}$$

Problem 5. Let $f(x)$

$$=\begin{cases}\dfrac{a^x-1}{x^n\sin x}\left(\dfrac{b\sin x-\sin bx}{\cos x-\cos bx}\right)^n & x>0\\[2ex] \dfrac{a^x\sin bx-b^x\sin ax}{\tan bx-\tan ax} & x<0\end{cases}$$

be continuous at $x=0$ $(a,b>0, b\neq 1, a\neq b)$. Obtain $f(0)$ and a relation between a, b and n.

Solution

$$f(0^+)\ \lim_{h\to 0}f(0+h)=\lim_{h\to 0}\frac{a^h-1}{\sinh.h^n}\left[\frac{b\sin h-\sin bh}{\cos h-\cos bh}\right]^n$$

$$==\lim_{h\to 0}\frac{a^h-1}{h}\ \frac{h}{\sin h}$$

$$\left[\left(\frac{b\sin(h)-\sin(bh)}{h.h^2}\right)\frac{h^2}{\cos h-\cos bh}\right]^n$$

$= \ln a\cdot l^n$ (say)

$$\text{where } l=\lim_{h\to 0}\frac{b\sin h-\sin bh}{h^3}\cdot\frac{h^2}{\cos h-\cos bh}$$

$$=\lim_{h\to 0}\left[\frac{b(\sin h-h)-(\sin bh-bh)}{h^3}\right]$$

$$.\left[b^2\left(\frac{1-\cos bh}{b^2h^2}\right)-\left(\frac{1-\cos h}{h^2}\right)\right]^{-1}$$

$$=\lim_{h\to 0}\left[b\left(\frac{\sin h-h}{h^3}\right)-\left(\frac{\sin bh-bh}{b^3h^3}\right)b^3\right]$$

$$\cdot\left[\left(\frac{b^2}{2}-\frac{1}{2}\right)^{-1}\right]$$

$$=\left[b\left(-\frac{1}{6}\right)-b^3\left(-\frac{1}{6}\right)\right]\cdot\frac{2}{(b^2-1)}$$

$$\left[\text{using }\lim_{h\to 0}\frac{\sin x-x}{x^3}=-\frac{1}{6}\right]$$

$$=\frac{b(b^2-1)}{6}\cdot\frac{2}{(b^2-1)}=\left(\frac{b}{3}\right)$$

Hence $f(0^+)=\ln a\cdot\left(\frac{b}{3}\right)^n$

Now $f(0^-)=\lim_{h\to 0}f(0-h)$

$$=\lim_{h\to 0}\frac{a^{-h}\sin(-bh)-b^{-h}\sin(-ah)}{\tan(-bh)-\tan(-ah)}$$

(multiply D^r & N^r by $a^h\cdot b^h$)

$$=\lim_{h\to 0}\frac{a^h\sin ah-b^h\sin bh}{a^h.b^h[\tan ah-\tan bh]}$$

$$=\lim_{h\to 0}\frac{(1+h\ln a.....)(ah....)-(1+h\ln b)(bh...)}{(ah...)-(bh....)}$$

$$=\lim_{h\to 0}\frac{(a-b)h+......}{(a-h)h+......}=1$$

$\therefore$ If $f(x)$ is continuous at $x=0$, then $f(0)=1$.

Also the continuity relationship between a, b and n should be $\ln a\cdot\left(\frac{b}{3}\right)^n=1$.

Problem 6. If $f(x)$

$$=\frac{\sin 3x\quad A\sin 2x\quad B\sin x}{x^5}\ (x\neq 0)$$

is continuous at $x=0$, then find $f(0)$.

Solution We have

$$f(0)=\lim_{x\to 0}\frac{1}{x^5}\left(\left[3x-\frac{(3x)^3}{3!}+\frac{(3x)^5}{5!}-......\right]\right.$$

$$+A\left[2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-.....\right]$$

$$\left.+B\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right]\right)\quad ...(1)$$

Now $f(x)$ is continuous at $x=0$, so we must have

$$2A+3+B=0 \qquad ...(2)$$

and $$\frac{27}{6}+\frac{8A}{6}+\frac{B}{6}=0$$

$$\Rightarrow\quad 8A+B=-27 \qquad ...(3)$$

∴ On solving (2) and (3), we get

$$A = -4,\ B = 5$$

Hence, $f(0) = \frac{3^5}{5!} + \frac{A.2^5}{5!} + \frac{B}{5!} = 1.$

Alternative:

We have f (0)

$$= \lim_{x\to 0} \frac{3\sin x - 4\sin^3 x + 2A\sin x\cos x + B\sin x}{x^5}$$

For limit to exist $3 + 2A + B = 0$ ...(4)

$$= \lim_{x\to 0} \frac{3\sin x - 4\sin^3 x + 2A\sin x\cos x - (3+2A)\sin x}{x^5}$$

$$= \lim_{x\to 0} \frac{\sin x}{x}\left[\frac{3 - 4\sin^2 x + 2A\cos x - 3 - 2A}{x^4}\right]$$

$$= \lim_{x\to 0} \frac{-2A(1-\cos x) - 4\sin^2 x}{x^4}$$

$$= \lim_{x\to 0} \frac{-4A\sin^2 \frac{x}{2} - 4\sin^2 x}{x^4}$$

$$= \lim_{x\to 0} \frac{-4A\sin^2 \frac{x}{2} - 16\sin^2 \frac{x}{2}\cos^2 \frac{x}{2}}{x^4}$$

$$= -\lim_{x\to 0} \frac{\sin^2 \frac{x}{2}}{\frac{x^4}{4}}\left[\frac{A + 4\cos^2 \frac{x}{2}}{x^2}\right]$$

∴ $A + 4 = 0 \Rightarrow A = -4 \Rightarrow B = 5$

Also $f(0) = 1$.

Problem 7. The function

$f(x) = \frac{e^{2x} - 1 - x\left(e^{2x} + 1\right)}{x^3}$ is not defined at $x = 0$.

What should be the value of $f(x)$ so that $f(x)$ is continuous at $x = 0$?

Solution $\ell = \lim_{x\to 0} \frac{e^{2x} - 1 - x\,(e^{2x} + 1)}{x^3}$

Put $x = 3t = \lim_{t\to 0} \frac{e^{6t} - 1 - 3t\,(e^{6t} + 1)}{27\,t^3}$

$$= \lim_{t\to 0} \frac{(e^{2t} - 1)^3 + 3e^{2t}\,(e^{2t} - 1) - 3t\,(e^{6t} + 1)}{27\,t^3}$$

$$= \lim_{t\to 0} \frac{(e^{2t}-1)^3 + 3e^{2t}[e^{2t}-1-t(e^{2t}+1)] + 3t[e^{2t}(e^{2t}+1) - e^{6t} - 1]}{27\,t^3}$$

$$= \lim_{t\to 0} \frac{(e^{2t} - 1)^3}{27\,t^3} + \frac{1}{9}\lim_{t\to 0} e^{2t}\times\ell$$

$$- \lim_{t\to 0} \frac{1}{9\,t^2}\,(e^{2t} - 1)\,(e^{4t} - 1)$$

$$\Rightarrow \frac{8\,\ell}{9} = \frac{8}{27} - \frac{8}{9}\lim_{t\to 0}\left(\frac{e^{2t} - 1}{2\,t}\right)\times\left(\frac{e^{4t} - 1}{4\,t}\right)$$

$$\Rightarrow \ell = \frac{1}{3} - 1 \Rightarrow \ell = -\frac{2}{3}.$$

The value of $f(0) = \ell = -\frac{2}{3}$

Problem 8. Let $f(x) = \text{cosec}\,2x + \text{cosec}\,2^2 x + \text{cosec}\,2^3 x + \ldots\ldots\ldots \text{cosec}\,2^n x,\ x \in (0, \pi/2)$

and $g(x) = f(x) + \cot 2^n x$

If

$$H(x) = \begin{cases} (\cos x)^{g(x)} + (\sec x)^{\text{cosec}\,x} & \text{if } x > 0 \\ p & \text{if } x = 0 \\ \dfrac{e^x + e^{-x} - 2\cos x}{x\sin x} & \text{if } x < 0 \end{cases}$$

find the value of p, if possible to make the function H (x) continuous at x = 0.

Solution $f(x) = \text{cosec}\,2x + \text{cosec}\,2^2x + \ldots. + \text{cosec}\,2^n x$

Now $\text{cosec}\,2x = \frac{1}{\sin 2x} = \frac{\sin(2x - x)}{\sin x\sin 2x} = \cot x - \cot 2x$

Similarly $\text{cosec}\,2^2 x = \cot 2x - \cot 2^2 x$

$\text{cosec}\,2^3 x = \cot 2^2 x - \cot 2^3 x$

⋮

$\text{cosec}\,2^n x = \cot 2^{n-1} x - \cot 2^n x$

$\Rightarrow f(x) = \cot x - \cot 2^n x$

∴ $g(x) = f(x) + \cot 2^n x = \cot x$

Now $\lim_{h\to 0} H(0 + h) = \lim_{h\to 0} ((\cos h)^{\cot h} + (\sec h)^{\text{cosech}})$

$$= e^{\lim_{h\to 0} \coth(\cos h - 1)} + e^{\lim_{h\to 0} \text{cosech}\,(\sec h - 1)}$$

$$= 1 + 1 = 2 \quad ...(1)$$

$$H(0 - h) = \lim_{h\to 0} \frac{e^{-h} + e^h - 2\cosh h}{h\,\text{sec}\,h}$$

$$= \lim_{h\to 0}\left\{\frac{e^h + e^{-h} - 2}{h^2} + \frac{2(1 - \cos h)}{h^2}\right\} = 2 \quad ...(2)$$

From (1) and (2) H (x) will be continuous if p = 2.

Problem 9. Check continuity of the function

$$f(x)=\lim_{n\to\infty}\frac{\cos\pi x - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}$$

Solution We have

$$f(x)=\begin{cases}\dfrac{\cos\pi x-0}{1+x-0}, & |x|<1 \quad \left[\because \lim\limits_{n\to\infty} x^{2n}=0 \text{ if } |x|<1\right]\\ \dfrac{\cos\pi x-\sin(x-1)}{1+1+1}, & |x|=1\\ \lim\limits_{n\to\infty}\dfrac{\dfrac{\cos\pi x}{x^{2n}}-\sin(x-1)}{\dfrac{1}{x^{2n}}+x-1}=\dfrac{-\sin(x-1)}{x-1}, & |x|>1\end{cases}$$

i.e. $$f(x)=\begin{cases}\cos\pi x, & |x|<1\\ -1+\sin 2, & x=-1\\ -1, & x=1\\ \dfrac{-\sin(x-1)}{x-1}, & |x|>1\end{cases}$$

The above function may have discontinuities only at $x=\pm 1$.

At $x=-1$, we have

$$\lim_{h\to 0} f(-1+h)=\lim_{h\to 0}\cos\pi(-1+h)=-1.$$

$f(-1)=-1+\sin 2$.

This implies that f(x) is discontinuous at $x=-1$.

At $x=1$, we have

$$\lim_{h\to 0} f(1+h)=\lim_{h\to 0}\frac{-\sin(1+h-1)}{1+h-1}=\lim_{h\to 0}\frac{-\sin h}{h}=-1.$$

$$\lim_{h\to 0} f(1-h)=\lim_{h\to 0}\cos\pi(1-h)=-1.$$

$f(1)=-1$

Thus, f(x) is discontinuous at $x=1$.

Hence, f(x) is continuous on $x\in R-\{-1\}$.

Problem 10. If $g(x)=\lim\limits_{m\to\infty}\dfrac{x^m f(1)+h(x)+1}{2x^m+3x+3}$ is continuous at $x=1$ and $g(1)=\lim\limits_{x\to 1}\{\ln(ex)\}^{2/\ln x}$, then find the value of $2g(1)+2f(1)-h(1)$; assume that f(x) and h(x) are continuous at $x=1$.

Solution Here,

$$g(1)=\lim_{x\to 1}\{\ln e+\ln x\}^{2/\ln x}$$

$$=\lim_{x\to 1}\{1+\ln x\}^{2/\ln x}$$

$$=e^{\lim\limits_{x\to 1}\ln x\frac{2}{\ln x}}$$

$$g(1)=e^2 \qquad ...(1)$$

Now, $$\lim_{x\to 1^-} g(x)=\lim_{x\to 1^-}\lim_{m\to\infty}\left\{\frac{x^m f(1)+h(x)+1}{2x^m+3x+3}\right\}$$

$$=\frac{h(1)+1}{3+3} \qquad \{\text{since } x<1\Rightarrow \lim_{m\to\infty} x^m=0\}$$

$$\therefore \lim_{x\to 1^-} g(x)=\frac{h(1)+1}{6} \qquad ...(2)$$

And, $$\lim_{x\to 1^+} g(x)=\lim_{x\to 1^+}\lim_{m\to\infty}\left\{\frac{x^m f(1)+h(x)+1}{2x^m+3x+3}\right\}$$

$$=\lim_{x\to 1^+}\lim_{m\to\infty}\frac{f(1)+h(x)/x^m+1/x^m}{2+3/x^{m-1}+3/x^m}=\frac{f(1)}{2}$$

$$\therefore \lim_{x\to 1^+} g(x)=\frac{f(1)}{2} \qquad ...(3)$$

As g(x) is continuous at $x=1$, from (1), (2) and (3)

$$e^2=\frac{h(1)+1}{6}=\frac{f(1)}{2}$$

$\Rightarrow$ $h(1)=6e^2-1$ and $f(1)=2e^2$

$\Rightarrow$ $2g(1)+2f(1)-h(1)=2e^2+4e^2-6e^2+1=1$

$\therefore$ $2g(1)+2f(1)-h(1)=1.$

Problem 11. Prove that $f(x)=[\tan x]+\sqrt{\tan x-[\tan x]}$. (where [.] denotes greatest integer function) is continuous in $\left[0,\dfrac{\pi}{2}\right)$.

Solution $f(x)=[\tan x]+\sqrt{\tan x-[\tan x]}$

Let $g(u)=[u]+\sqrt{u-[u]}$,

Then $f(x)=g(u(x))$ where $u(x)=\tan x\ge 0$

We discuss continuity of g(u) for $u=a\in N$.

LHL at $u=a$:

$$\lim_{u\to a^-} g(u)=\lim_{h\to 0}[a-h]+\sqrt{a-h-[a-h]}$$

$$=\lim_{h\to 0}(a-1)+\sqrt{a-h-(a-1)}$$

$$=a-1+1=a$$

Now RHL at $u=a$:

$$\lim_{u\to a^+} g(u)=\lim_{h\to 0}[a+h]+\sqrt{a+h-[a+h]}$$

$$=\lim_{h\to 0} a+\sqrt{a+h-a}$$

$$=a,$$

and $g(a)=[a]+\sqrt{a-[a]}=a$ as $a\in N$.

So, g(u) is continuous $\forall\, a\in N$, now g(u) is clearly continuous in $(a-1, a)\ \forall\, a\in N$.

Hence g(u) is continuous in $[0, \infty)$.
Now $u(x) = \tan x$ is continuous in $[0, \pi/2)$.
So, $f(x) = g\{u(x)\}$ is continuous in $[0, \pi/2)$.

Problem 12. If $f(x) = \frac{x}{(1+x)} + \frac{x}{(1+x)(1+2x)} + \frac{x}{(1+2x)(1+3x)} + ...$ to infinity, then examine the continuity of f at x = 0.

Solution For $x \neq 0$, sum of n terms of the series

$$\left(1-\frac{1}{1+x}\right)+\left(\frac{1}{1+x}-\frac{1}{1+2x}\right)+\left(\frac{1}{1+2x}-\frac{1}{1+3x}\right)+...+\left(\frac{1}{1+(n-1)x}-\frac{1}{1+nx}\right)$$

$$= 1 - \frac{1}{1+nx}.$$

$$\therefore \quad f(x) = \lim_{n\to\infty}\left(1-\frac{1}{1+nx}\right) = 1 - 0 = 1.$$

and $f(0) = 0$

$$\therefore \quad f(x) = \begin{cases} 1, & x \neq 0 \\ x, & x = 0 \end{cases}$$

Clearly f(x) is discontinuous at x = 0.

Problem 13. Show that the function $f : R \to R$ defined by $f(x) = \frac{1}{1 + e^{1/(\sin n!\,\pi x)}}$ can be made discontinuous at any rational point in the interval [0, 1] by a proper choice of n.

Solution Let x be rational say p/q, where p and q are integers prime to each other. Then taking n = q, we see that

$$n!\,\pi x = q!\,.\,\pi\,.\,(p/q)$$

is an integral multiple of π and therefore $\sin n!\,\pi x = 0$ but $\cos n!\,\pi x = 1$ or -1 according as n! x is an even or an odd integer. Now

$$\sin n!\,\pi(x \pm h) = \sin n!\,\pi x \cos n!\,\pi h \pm \cos(n!\,\pi x)\sin(n!\,\pi h)$$

$$= \pm \sin n!\,\pi h \text{ or } \mp \sin(n!\,\pi h) \qquad ...(1)$$

according as n! x is even or odd.

(i) Let n! x be an even integer.

Then $f(x^+) = \lim_{h\to 0}\frac{1}{1+e^{1/\sin\{n!\pi(x+h)\}}}$

$$= \lim_{h\to 0}\frac{1}{1+e^{1/\sin(n!\,\pi h)}} = \frac{1}{1+e^{\infty}} = 0 \quad \text{[using (1)]}$$

and $f(x^-) = \lim_{h\to 0}\frac{1}{1+e^{1/\sin\{n!\pi(x-h)\}}}$

$$= \lim_{h\to 0}\frac{1}{1+e^{-1/\sin(n!\pi h)}} \quad \text{[using (1)]}$$

$$= \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

(ii) Let n! x be an odd integer. Then

$$f(x^+) = \lim_{h\to 0}\frac{1}{1+e^{-1/\sin(n!\pi h)}} = 1$$

and $f(x^-) = \lim_{h\to 0}\frac{1}{1+e^{1/\sin(n!\pi h)}} = 0$.

Hence $f(x^+) \neq f(x^-)$ at any rational point x.

Problem 14. Show that the function $f : R \to R$ defined by

$$f(x) = \lim_{t\to\infty}\frac{(1+\sin\pi x)^t - 1}{(1+\sin\pi x)^t + 1}$$

has discontinuity of first kind at the points

x = 0, 1, 2, ..., n .... .

Solution At x = 0, 1, 2, 3, ... , n .... , we have

$\sin \pi x = 0$, so that $f(x) = \lim_{t\to\infty}\frac{(1+0)^t - 1}{(1+0)^t + 1} = 0$

at these values.

Now, if $2m < x < 2m + 1$ (m being an integer), then $\sin \pi x$ is positive. Hence for such values of x, we have

$$f(x) = \lim_{t\to\infty}\frac{1 - 1/(1+\sin\pi x)^t}{1 + 1/(1+\sin\pi x)^t} = \frac{1-(1/\infty)}{1+(1/\infty)} = 1$$

And if $2m + 1 < x < 2m + 2$. $\sin \pi x$ is negative and so

$$\lim_{t\to\infty}(1+\sin\pi x)^t = 0.$$

$$\therefore \quad f(x) = \frac{0-1}{0+1} = -1 \text{ for these values of x.}$$

Hence if x an even integer, then

$f(x) = 0$, $f(x^+) = 1$ and $f(x^-) = -1$,

and if x is an odd integer, then

$f(x) = 0$, $f(x^+) = -1$ and $f(x^-) = 1$.

Hence f has discontinuities of the first kind at

x = 0, 1, 2, ... , n , ... ,

Problem 15. Discuss the nature of the discontinuity of the function f defined by

$$f(x) = \lim_{n\to\infty}\frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}} \text{ at } x = 1.$$

Show that f(0) and $f(\pi/2)$ differ in sign and explain still why f does not vanish in $[0, \pi/2]$.

Solution We shall first of all obtain an expression for f in $[0, \pi/2]$ in a form free from limits.

If $0 \le x < 1$, then $x^{2n} \to 0$ as $n \to \infty$, and if $x > 1$, then $x^{-2n} \to 0$ as $n \to \infty$, therefore we have :

If $0 \le x < 1$, then $f(x) = \lim_{n\to\infty} \dfrac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$

$= \log(2+x)$ ...(1)

If $x = 1$, then $f(x) = \lim_{n\to\infty} \dfrac{\log(3) - 1^{2n}\sin 1}{+1^{2n}}$

$= \dfrac{1}{2}(\log 3 - \sin 1)$ ...(2)

If $x > 1$, then $f(x) = \lim_{n\to\infty} \dfrac{x^{-2n}\log(2+x) - \sin x}{x^{-2n}+1}$

$= -\sin x$ ...(3)

From (1), (2) and (3), we have

$$f(x) = \begin{cases} \log(2+x) & \text{if } 0 \le x < 1 \\ \frac{1}{2}(\log 3 - \sin 1), & \text{if } x = 1 \\ -\sin x & \text{if } x > 1 \end{cases} \quad ...(4)$$

$f(1^-) = \lim_{h\to 0^+} f(1-h) = \lim_{h\to 0^+} \log(3-h) = \log 3$

$f(1^+) = \lim_{h\to 0^+} f(1+h) = \lim_{h\to 0^+} -\sin(1+h) = -\sin 1.$

We find that $f(1^-)$ and $f(1^+)$ both exist but are unequal; also neither of them is equal to f(1). Therefore f has a discontinuity at x = 1.

From (4), we find that

$f(0) = \log 2 > 0$, $f(\pi/2) = -\sin \pi/2 = -1$,

so that f(0) and $f(\pi/2)$ are of opposite signs.

Again, from (4) it is clear that f does not vanish anywhere in $[0, \pi/2]$.

The function f is not continuous in $[0, \pi/2]$, the point x = 1 being a point of discontinuity. This explains as to why f does not vanish anywhere in $[0, \pi/2]$ even though f(0) and $f(\pi/2)$ are of opposite signs.

The hypothesis as well as the conclusion of the Intermediate Value Theorem are not satisfied for the function f in $[0, \pi/2]$.

Things to Remember

1. A function f(x) is said to be continuous at $x = a$, if $\lim_{x\to a} f(x) = f(a)$.

2. A function f(x) is said to be continuous at the left end point $x = a$ if, $f(a) = \lim_{x\to a^+} f(x)$ and f(x) is said to be continuous at the right end point x = b if, $f(b) = \lim_{x\to b^-} f(x)$

3. $F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x = a \end{cases}$

The function F is continuous is called the continuous extension of f to x = a, provided $f(x) = L$ exists.

4. A function f is said to be continuous in an open interval (a , b) if f is continuous at each and every point lying in the interval (a , b) .

5. A function f is said to be continuous in a closed interval [a, b] if :

(i) f is continuous in the open interval (a , b)

(ii) f is right continuous at 'a' i.e. $\lim_{x\to a^+} f(x) = f(a)$.

(iii) f is left continuous at 'b' i.e. $\lim_{x\to b^-} f(x) = f(b)$.

6. We use the term suspicious point for a number c where

(i) The definition of the function changes or domain of f splits,

(ii) Substitution of x = c causes division by 0 in the function.

7. If both the one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, but the conditions of continuity are not satisfied. Then the function f(x) is said to have a discontinuity of the first kind at the point a.

8. A function f(x) having a finite number of discontinuities of first kind in a given interval is called sectionally or piecewise continuous function.

9. The function f(x) is said to have discontinuity of the second kind at x = a, if atleast one of the one-sided limits (L.H.L. or R.H.L.) at the point x = a does not exist or equals to infinity.

10. A function is said to have a removable discontinuity at $x = a$, if $\lim_{x\to a} f(x)$ exists but is not equal to f(a). In this case we can redefine the function such that $\lim_{x\to a} f(x) = f(a)$ and make it continuous at $x = a$.

11. A function is said to have a missing point discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ exists while the function is undefined at $x = a$.

12. A function is said to have an isolated point discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ exists and the function is defined at $x = a$, but they are unequal.

13. A function is said to have an irremovable discontinuity at $x = a$, if $\lim_{x\to a} f(x)$ does not exist. In this case we cannot redefine the function such that $\lim_{x\to a} f(x) = f(a)$ and make it continuous at $x = a$.

14. A function is said to have a finite or jump discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ does not exist since the left hand limit and the right hand limit are unequal, but the one-sided limits do exist.

15. If $x = a$ is a point of finite discontinuity of the function $f(x)$, then the graph of this function undergoes a jump at $x = a$. The difference R.H.L. – L.H.L. i.e. $f(a^+) - f(a^-)$ is called the jump in the function at $x = a$.

16. The difference between the greatest and least of the three numbers $f(a^+)$, $f(a^-)$, $f(a)$ is the saltus or measure of discontinuity of the function at the point a.

17. A function is said to have an infinite discontinuity at $x = a$ if atleast one of the one-sided limits is infinite.

18. The concept of pole discontinuity is related with infinite limit. For a point $x = a$ to qualify as a pole of a function f, we must have $\lim_{x\to a} \frac{1}{f(x)} = 0$.

19. A function is said to have an oscillatory discontinuity at $x = a$ if atleast one of the one-sided limits does not exist because of too much oscillation in the values of the function.

20. If $f(x)$ and $g(x)$ are continuous at $x = a$, then the following functions are also continuous at $x = a$.

(i) $cf(x)$ is continuous at $x = a$, where c is any constant.

(ii) $f(x) \pm g(x)$ is continuous at $x = a$.

(iii) $f(x).\ g(x)$ is continuous at $x = a$.

(iv) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.

21. If $f(x)$ is continuous at $x = a$ and $g(x)$ is discontinuous at $x = a$, then we have the following results.

(i) Both the functions $f(x) + g(x)$ and $f(x) - g(x)$ are discontinuous at $x = a$.

(ii) $f(x).\ g(x)$ is not necessarily discontinuous at $x = a$.

(iii) $f(x)/g(x)$ is not necessarily discontinuous at $x = a$.

22. If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then we have the following results.

(i) The functions $f(x) + g(x)$ and $f(x) - g(x)$ are not necessarily discontinuous at $x = a$. However, atmost one of $f(x) + g(x)$ or $f(x) - g(x)$ can be continuous at $x = a$. That is, both of them cannot be continuous simultaneously at $x = a$.

(ii) $f(x).\ g(x)$ is not necessarily discontinuous at $x = a$. (iii) $f(x)/g(x)$ is not necessarily discontinuous at $x = a$.

23. If $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = f(a)$ then the composite function $(gof)(x)$ is continuous at $x = a$.

24. Let a function $f(x)$ be continuous at all points in the interval $[a, b]$, and let its range be the interval $[A, B]$ and further a function $g(x)$ is continuous in the interval $[A, B]$, then the composite function $(gof)(x)$ is continuous in the interval $[a, b]$.

25. If the function f is continuous everywhere and the function g is continuous everywhere, then the composition gof is continuous everywhere.

26. All polynomials, trigonometric functions, inverse trigonometric functions, exponential and logarithmic functions are continuous at all points in their domains.

27. If $f(x)$ is continuous, then $|f(x)|$ is also continuous.

28. A root of a continuous function is continuous, wherever it is defined. That is, the composition $h(x) = \sqrt[n]{g(x)} = [g(x)]^{1/n}$ of $f(x) = \sqrt[n]{x}$ and the function $g(x)$ is continuous at a if g is, assuming that $g(a) \geq 0$ if n is even (so that $\sqrt[n]{g(a)}$ is defined).

29. If the function $y = f(x)$ is defined, continuous and strictly monotonic on the interval I, then there exist a single valued inverse function $x = g(y)$ defined, continuous and also strictly monotonic in the range of the function $y = f(x)$.

30. Assume f is integrable on [a, x] for every x in [a, b] and let $A(x) = \int_a^x f(t)dt$. Then the integral A is continuous at each point of [a, b]. (At each endpoint we have one-sided continuity)

31. If a function f(x) is continuous at every point of a closed interval [a, b], then the function f(x) is bounded on this interval. Note that the boundedness of a function on the interval [a, b] means that there is a number K > 0 such that $|f(x)| < K$ for all $x \in [a, b]$.

32. If a function is continuous in a closed interval there exists atleast one point at which the function assumes the greatest value and atleast one point at which it assumes the least value on that interval.

33. If f is continuous at c and $f(c) \neq 0$, then there exists an interval $(c - \delta, c + \delta)$ around c such that f(x) has the sign of f(c) for every value of x in this interval.

34. If a function f is continuous on a closed interval [a, b] and the numbers f(a) and f(b) are different from zero and have opposite signs, then there is atleast one point c on the open interval (a, b) such that f(c) = 0.

35. If f is continuous on the closed interval [a, b] and k is any number between f(a) and f(b), then there is atleast one number c in [a, b] such that f(c) = k.

36. A function continuous on a closed interval [a, b] assumes all the intermediate values lying between its least and greatest values on [a, b].

37. A continuous function whose domain is closed must have a range in a closed interval but it is not necessary that if the domain is open then range is also open (range can be closed).

38. Let f be a continuous strictly increasing function on a closed interval [a, b] and let $\alpha = f(a)$, $\beta = f(b)$. Then (i) the range of f for the closed interval [a, b] is the closed interval $[\alpha, \beta]$, (ii) there exists a function x = g(y), the inverse of f, which is one-valued, strictly increasing and continuous on $[\alpha, \beta]$.

A strictly decreasing function f(x), continuous on [a, b] has an inverse, which is a strictly decreasing continuous function on $[\beta, \alpha]$ where $\alpha = f(a)$, $\beta = f(b)$.

39. Let a strictly increasing function y = f(x) map the closed interval [a, b] onto the closed interval $[\alpha, \beta]$, i.e. $f([a, b]) = [\alpha, \beta]$. Then f is continuous on [a, b].

40. Let f(x) be a function defined and bounded in the interval [a, b], then, if M and m are the bounds of f(x), the number M – m is called the span or oscillation of the function f(x) in the interval.

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. The function f(x)
$= \frac{1 - \cos x(\cos x(\cos 2x)^{1/2}(\cos 3x)^{1/3}}{x^2}$ is not defined at x = 0. If f(x) is continuous at x = 0 then f(0) equals

(A) 1 (B) 3
(C) 6 (D) –6

2. If $f(x) = \frac{x - e^x + \cos 2x}{x^2}$, $x \neq 0$ is continuous at x = 0, then

(A) $f(0) = \frac{5}{2}$ (B) $[f(0)] = -2$
(C) $\{f(0)\} = -0.5$ (D) $[f(0)].\{f(0)\} = -1.5$

3. If $f(x) = \begin{cases} |x^2 - 1| - 1 & , x \le 1 \\ |2x - 3| - |x - 2| & , x > 1 \end{cases}$, then it is continuous for

(A) R (B) R – {1}...

(A) R (B) R – {0}
(C) R – {1} (D) none of these

4. If $f(x) = [x] + \left[x + \frac{1}{3}\right] + \left[x + \frac{2}{3}\right]$ then number of points of discontinuity of f(x) in [–1, 1] is

(A) 5 (B) 4
(C) 7 (D) none

5. If $f(x) = \begin{cases} \frac{\sin\{x\}}{\{x\}}, & \{x\} \neq 0 \\ K, & \{x\} = 0 \end{cases}$, where [.] denotes fractional part function, then f(x) will be continuous

(A) if K = 0 (B) if K = sin 1
(C) if K = 1 (D) for no value of K

6. If f(x) is a continuous function from $R \to R$ and attains only irrational values, then $\sum_{r=1}^{100} f(r)$ is equal to

(A) $\sum_{r=1}^{99} f(r)$ (B) $\sum_{r=1}^{100} f(2r)$

(C) $\sum_{r=1}^{100} f(2r)+1$ (D) none of these

7. If $f(x) = \text{sgn}(\cos 2x - 2\sin x + 3)$, then f(x)
(A) is continuous over its domain
(B) has a missing point discontinuity
(C) has isolated point discontinuity
(D) irremovable discontinuity

8. Let $f(x) = (\sin x)^{\frac{1}{\pi - 2x}}$, $x \neq \pi/2$ if f(x) is continuous at $x = \pi/2$ then $f(\pi/2)$ is,
(A) e (B) 1
(C) 0 (D) none

9. If $f(x) = \frac{x+1}{x}$, the number of points of discontinuity of the composite function. y = f (f(f(x))) are
(A) 0 (B) 1
(C) 2 (D) 3

10. Let $f(x) = \begin{cases} \frac{a|x^2 - x - 2|}{2 + x - x^2}, & x < 2 \\ b, & x = 2 \\ \frac{x - [x]}{x - 2}, & x > 2 \end{cases}$

If f(x) is continuous at x = 2 (where [.] denotes greatest integer function) then (a, b) is
(A) (1, 1) (B) (1, 2)
(C) (2, 1) (D) (2, 2)

11. The function $f(x) = |2\,\text{sgn}\,2x| + 2$ has
(A) jump discontinuity
(B) removable discontinuity
(C) infinite discontinuity
(D) no discontinuity

12. If $g(x) = \lim_{x \to \infty} \left(\frac{x + x^{2n} \sin x}{1 + x^{4n}} \right)$ then
(A) g(x) is continuous at x = 1
(B) g(x) is discontinuous at x = 1
(C) limit does not exist at x = 0
(D) none of these

13. Let $f(x) = \frac{x^3}{4} - a\sin \pi x + 3, -4 \le x \le 4$. The value of f(x) is $\frac{1999}{199}$ for some $x \in [-4, 4]$ This statement is
(A) true (B) false
(C) true only if $a \ge 0$ (D) true only if $a \in [-4, 4]$

14. The ordered pair (a, b) such that

$$f(x) = \begin{cases} \frac{be^x - \cos x - x}{x^2}, & x > 0 \\ a, & x = 0 \\ 2\frac{\left(\tan^{-1}(e^x) - \frac{\pi}{4}\right)}{x}, & x < 0 \end{cases}$$

becomes continuous at x = 0 is
(A) (1, 1) (B) $\left(\frac{1}{2}, 1\right)$
(C) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (D) not possible

15. The function $f(x) = \begin{cases} 2x + 1, & x \in Q \\ x^2 - 2x + 5, & x \notin Q \end{cases}$ is
(A) continuous no where
(B) continuous at every rational point
(C) continuous at irrational points only
(D) continuous exactly at one point

16. Let [x] denotes the greatest integer less than or equal to x. If $f(x) = [x \cos x]$, the f(x) is
(A) continuous at x = 0
(B) continuous in (–1, 0)
(C) discontinuous at x = 1
(D) continuous in (–1, 1)

17. If $f(x) = \frac{\sin^4\left(\frac{1}{x}\right) - \sin^2\left(\frac{1}{x}\right) + 1}{\cos^4\left(\frac{1}{x}\right) - \cos^2\left(\frac{1}{x}\right) + 1}$ is to be made continuous at x = 0, then f(0) should be equal to
(A) 0 (B) 1
(C) 1/3 (D) 1/2

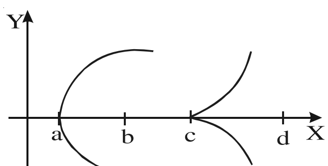
18. If $f(x) = \lim_{n \to \infty} (1 - \sin^2 x)^n$, $n \in N$, then
(A) $f\left(\frac{\pi}{4}\right)$ is nearly zero
(B) $f\left(\frac{\pi}{4}\right)$ is equal to $\frac{1}{2}$
(C) f(x) is discontinuous at infinite number of points
(D) f(x) is a periodic

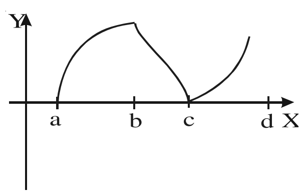
19. Let $f(x)=\begin{bmatrix} a\sin^{2n}x & \text{for } x\ge 0 \text{ and } n\to\infty \\ b\cos^{2m}x-1 & \text{for } x<0 \text{ and } m\to\infty \end{bmatrix}$ then

(A) $f(0^-)\neq f(0^+)$ (B) $f(0^+)\neq f(0)$
(C) $f(0^-)$ - $f(0)$ (D) f is continuous at $x=0$

20. f is a continuous function in [a, b]; g is a continuous function in [b, c]
A function h(x) is defined as
$h(x) = f(x)$ for $x \in [a, b)$
$= g(x)$ for $x \in (b, c]$
if $f(b) = g(b)$, then
(A) h(x) has a removable discontinuity at x = b.
(B) h(x) may or may not be continuous in [a, b]
(C) $h(b^-) = g(b^+)$ and $h(b^+) = f(b^-)$
(D) $h(b^+) = g(b^-)$ and $h(b^-) = f(b^+)$

21. If graph of |y| = f(x) and y |f(x)| is as shown below





Then number of points of discontinuity of f(x) in [a, d] is
(A) 1 (B) 2
(C) 3 (D) 4

22. If $x \in R^+$ and $n \in N$, we can uniquely write $x = mn + r$ where $m \in W$ and $0 \le r < n$. We define x mod n = r. For example 10.3 mod 3 = 1.3. The number of points of discontinuity of the function $f(x) = (x \bmod 2)^2 + (x \bmod 4)$ in the interval $0 < x < 9$ is
(A) 0 (B) 2
(C) 4 (D) none

23. If graph of the function y = f(x) is continuous and passes through point (3, 1) then $\lim_{x\to 3}\frac{\ell n(3f(x)-2)}{2(1-f(x))}$ is equal
(A) $\frac{3}{2}$ (B) $\frac{1}{2}$
(C) $-\frac{3}{2}$ (D) $-\frac{1}{2}$

24. Let f(x) = the highest power of $(u^{x^2} + u^2 + 2u + 3)$. Then at $x = \sqrt{2}$, f(x) is
(A) continuous (B) $\lim_{x\to\sqrt{2}} f(x) = -2$
(C) discontinuous (D) none of these

25. Let f(x) = [sin x + cos x], $0 < x < 2\pi$, (where [.] denotes the greatest integer function) Then the number of points of discontinuity of f(x) is
(A) 6 (B) 5
(C) 4 (D) 3

26. If $\alpha, \beta (\alpha < \beta)$ are the points of discontinuity of the function f(f(f(x))) where $f(x) = \frac{1}{1-x}$, then the set of values of 'a' for which the points (α, β) and (a, a^2) lie on the same side of the line $x + 2y - 3 = 0$, is
(A) $\left(-\frac{3}{2},1\right)$ (B) $\left[-\frac{3}{2},1\right]$
(C) $[1,\infty)$ (D) $\left(-\infty,-\frac{3}{2}\right]$

27. Let $f : (0, \infty) \to R$ be a continuous function such $2 \le \frac{f(x)}{x^2} \le 3$ for all $x > 0$. Then with certainity f(c) = c for atleast one point 'c' belonging to the interval (s)
(A) $\left(\frac{1}{2},1\right)$ (B) $\left(\frac{1}{3},\frac{1}{2}\right)$
(C) $\left(\frac{1}{8},\frac{1}{4}\right)$ (D) $\left(\frac{1}{4},1\right)$

28. If $f(x) = \text{sgn}(\sin^2 x - \sin x - 1)$ has exactly four points of discontinuity for $x \in (0, n\pi)$, $n \in N$ then
(A) the minimum value of n is 5
(B) the maximum value of n is 6
(C) there are exactly two possible values of n
(D) none of these

29. $f(x) = \lim_{n\to\infty} \sin^{2n}(\pi x) + \left[x + \frac{1}{2}\right]$, where [.] denotes the greatest integer function is
(A) Continuous at x = 1 but discontinous at $x=\frac{3}{2}$
(B) Continuous at x = 1 and $x = \frac{3}{2}$

(C) Discontinuous at x = 1 and $x = \frac{3}{2}$

(D) Discontinous at x = 1 but continuous at $x = \frac{3}{2}$

30. If f(x) is a continuous function ∀ x ∈ R and the range of f(x) = $(2, \sqrt{26})$ and $g(x) = \left[\frac{f(x)}{a}\right]$ is continuous ∀ x ∈ R (where [.] denotes the greatest integer function), then the least positive integral value of a is

(A) 2 (B) 3
(C) 6 (D) 5

31. Let $f(x) = \begin{cases} \frac{axe^x + b\sin x}{x^3} & , x < 0 \\ c\cos\pi\{x\} & , x \ge 0 \end{cases}$, where {.} represents fractional part function, if f(x) is continuous at x = 0, then the value of c is :

(A) $\frac{4}{3}$ (B) $\frac{2}{3}$
(C) $\frac{1}{3}$ (D) – 1

32. Let f R → R such that f(x) is continuous and attains only rational values at all real x and f(3)=4. If a_1, a_2, a_3, a_4, a_5 are in H.P., then $\sum_{n=1}^{n} a_r a_{r+1}$ is

(A) f(5). a_1a_5 (B) f(3). a_4a_5
(C) f(3). a_1a_2 (D) f(2). a_1a_3

33. If $f(x) = \begin{cases} 2x - 1 & , -2 \le x < 0 \\ x + 2 & , 0 \le x \le 2 \end{cases}$ and $g(x) = \begin{cases} [x] & , -4 \le x < -2 \\ x + 2 & , -2 \le x \le 4 \end{cases}$ then

(A) $\lim_{x \to -2^+} f(g(x)) = 2$

(B) f(g(x)) is discontinuous at x = –2

(C) f(g(x)) is not defined at x = 2

(D) none of these

34. The function defined by $f(x) = [x^2 + e^{1/(2-x)}]^{-1}$ when x > 2 and f(x) = k when x = 2 is continuous in interval [2, ∞). Than k is equal to

(A) 0 (B) 1/4
(C) –1/4 (D) None

35. Let $f(x) = \begin{cases} \sin^2 x & , x \text{ rational} \\ -\sin^2 x & , x \text{ irrational} \end{cases}$.

Then set of points where f(x) is continuous -

(A) $\left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

(B) null set

(C) {nπ, n ∈ I}

(D) set of all rational numbers

36. If the graph of the continuous function y = f(x) passes through (a, 0), then

$$\lim_{x \to a} \frac{\ln(1 + 6f^2(x) - 3f(x))}{3f(x)}$$ is equal to

(A) 1 (B) 0
(C) – 1 (D) None

37. Let f be a continuous function on R such that $f\left(\frac{1}{4^n}\right) = (\sin e^n)\, e^{-n^2} + \frac{n^2}{n^2 + 1}$. Then f(0) is equal to

(A) 0 (B) 1
(C) 2 (D) none

38. Let f(x) = [tan x[cot x]], $x \in \left[\frac{\pi}{12}, \frac{\pi}{2}\right)$ (where [.] represents the greatest integer function) then the number of points, where f(x) is not continuous is

(A) one (B) zero
(C) three (D) infinite

39. If f : R → R and g : R → R such that

$f(x) = \begin{cases} 0 \text{ if } x \text{ rational} \\ 1 \text{ if } x \text{ irrational} \end{cases}$; $g(x) = \begin{cases} 0 \text{ if } x \text{ irrational} \\ 1 \text{ if } x \text{ rational} \end{cases}$

then

(A) f + g is discontinuous

(B) f + g is continuous at rational only

(C) f + g is continuous everywhere

(D) f + g is continuous for irrationals only

40. Let $f(x) = g(x)\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ and x ≠ 0, where g is a continuous function. Then $\lim_{x \to 0} f(x)$ exists if

(A) g(x) is any polynomial

(B) g(x) = x + 4

(C) $g(x) = x^2$

(D) $g(x) = 2 + 3x + 4x^2$

41. The point of discontinuity of the function

$$f(x) = \lim_{n \to \infty} \frac{(2\sin x)^{2n}}{3^n - (2\cos x)^{2n}}$$ is

(A) $n\pi \pm \pi/3;\ n \in I$ (B) $n\pi \pm \pi/6;\ n \in I$
(C) $2n\in;\ n \in I$ (D) none of these

42. The set of all points of discontinuity of the function $f(x) = \left(\frac{\tan x \log x}{1-\cos 4x}\right)$ contains

(A) $\left\{\frac{n\pi}{2}; n \in I\right\}$ (B) $\left\{\frac{n\pi}{2}; n \in Q\right\}$

(C) $(-\infty, 0] \cup \left\{\frac{n\pi}{2}; n \in N\right\}$

(D) none of these

43. Let $f(x) = \sin x$ if x is rational and $f(x) = 1 - 2\cos x$ if x is irrational then
(A) f(x) is no where continuous
(B) f(x) is continuous at one point only
(C) f(x) is injective
(D) f(x) is continuous at infinite no. of points

44. The function f(x) is defined by

$$f(x)=\begin{cases}\log_{(4x-3)}(x^2-2x+5) & \text{if } \frac{3}{4}<x<1 \,\&\, x>1\\ 4 & \text{if } x=1\end{cases}$$

(A) is continuous at x = 1
(B) is discontinuous at x = 1 since $f(1^+)$ does not exist though $f(1^-)$ exists
(C) is discontinuous at x = 1 since $f(1^-)$ does not exist though $f(1^+)$ exists
(D) is discontinuous since neither $f(1^-)$ nor $f(1^+)$ exists.

45. The function f defined by

$$f(x) = \lim_{t\to\infty}\left\{\frac{(1+\sin \pi x)^t - 1}{(1+\sin \pi x)^t + 1}\right\} \text{ is}$$

(A) everywhere continuous
(B) discontinuous at all integer values of x
(C) continuous at x = 0
(D) none of these

46. The function $f : R/\{0\} \to R$ given by $f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}$ can be made continuous at $x = 0$ by defining f(0) as
(A) 2 (B) –1
(C) 0 (D) 1

47. If $f(x)=\begin{cases}\frac{1-|x|}{1+x}, & x \neq -1\\ 1, & x=-1\end{cases}$, then f([2x]) is where [ ] represent greatest integer function
(A) continuous at x = –1
(B) continuous at x = 0
(C) discontinuous at x = 1/2
(D) all of these

48. Let $f(x)=\begin{cases}-1, & x\le 0\\ 0, & x=0\\ 1, & x>0\end{cases}$ and $g(x) = \sin x + \cos x$, then points of discontinuity of $f\{g(x)\}$ in $(0, 2\pi)$ is

(A) $\left\{\frac{\pi}{2}, \frac{3\pi}{4}\right\}$ (B) $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$

(C) $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$ (D) $\left\{\frac{5\pi}{4}, \frac{7\pi}{3}\right\}$

49. The value of 'a' for which the function $f(x)=\begin{cases}2x^3, & x\le 2\\ ax^2, & x>2\end{cases}$ $g(x)=\begin{cases}4\frac{\sin x}{x}, & x<0\\ a-2x, & x\ge 0\end{cases}$ and $h(x)=\begin{cases}\frac{x^2-a^2}{x-a}, & x\neq a\\ 8, & x=a\end{cases}$ are all continuous is
(A) 2 (B) 4
(C) no value of a exists (D) none of these

50. The set of all points of discontinuity of the function $f(x) = \left(\frac{\tan x \log x}{1-\cos 4x}\right)$ contains

(A) $\left\{\frac{n\pi}{2}, n \in I\right\}$ (B) $\left\{\frac{n\pi}{2}, n \in Q\right\}$

(C) $(-\infty, 0] \cup \left\{\frac{n\pi}{2}, n \in N\right\}$

(D) none of these

51. Let $f(x)=\begin{cases}0, & \text{if } x \text{ is rational}\\ 5x, & \text{if } x \text{ is irrational}\end{cases}$.

Then
(A) f (x) is continuous at every rational number.
(B) f (x) is continuous at every irrational number.
(C) f (x) is discontinuous everywhere.
(D) f (x) is continuous only at x = 0.

52. Let $R(x) = \frac{x^3 - 2x^2 - 9x + 18}{x^4 - 4}$

Which of the following statements describes the graph of y = R(x) :

(A) The graph has two vertical asymptotes
(B) The graph has two holes in it
(C) The graph has one hole and one vertical asymptote
(D) The graph has neither holes nor asymptotes

53. Given that $\prod_{n=1}^{n}\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$.

Let $f(x)=\begin{cases}\lim\limits_{n\to\infty}\sum\limits_{n=1}^{n}\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right), & x\in(0,\pi)-\left\{\frac{\pi}{2}\right\}\\ \frac{2}{\pi}, & x=\frac{\pi}{2}\end{cases}$

Then which one of the following alternative is true?
(A) f(x) has non-removable discontinuity of finite type at $x=\frac{\pi}{2}$.
(B) f(x) has missing point discontinuity at $x=\frac{\pi}{2}$.
(C) f(x) is continuous at $x=\frac{\pi}{2}$.
(D) f(x) has non-removable discontinuity of infinite type at $x=\frac{\pi}{2}$.

54. The value of f(0) such that $f(x)=\frac{2-\sqrt[4]{x^2+16}}{\cos 2x-1}$ is continuous at $x=0$ is :
(A) 1/64 (B) −1/64
(C) 1/32 (D) −1/32

55. Consider

$$f(x)=\left[\frac{2\left(\sin x-\sin^3 x\right)+\left|\sin x-\sin^3 x\right|}{2\left(\sin x-\sin^3 x\right)-\left|\sin x-\sin^3 x\right|}\right],$$

$x\neq\frac{\pi}{2}$ for $x\in(0,\pi)$

$f(\pi/2)=3$ where [.] denotes the greatest integer function then,
(A) f is continuous & differentiable at $x=\pi/2$
(B) f is continuous but not differentiable at $x=\pi/2$
(C) f is neither continuous nor differentiable at $x=\pi/2$
(D) none of these

56. Given $f(x)=b([x]^2+[x])+1$ for $x\geq -1$
$=[\sin(\pi(x+a))]$ for $x<-1$
where [x] denotes the integral part of x, then for what values of a, b the function is continuous at $x=-1$?
(A) $a=2n+(3/2)$; $b\in R$; $n\in I$
(B) $a=4n+2$; $b\in R$; $n\in I$
(C) $a=4n+(3/2)$; $b\in R^+$; $n\in I$
(D) $a=4n+1$; $b\in R^+$; $n\in I$

57. The function $f(x)=\begin{cases}[x]+\sqrt{x-[x]} & \text{for } x\geq 0\\ \sin x & \text{for } x<0\end{cases}$ is
(A) continuous only for all non-negative integers
(B) continuous only for all positive integers
(C) discontinuous only for all negative integers
(D) cont. for all real numbers .

58. The function $f(x)=\frac{4-x^2}{\left|4x-x^3\right|}$ is :
(A) discontinuous at only one point
(B) discontinuous at exactly two points
(C) discontinuous at exactly three points
(D) none

59. Let $f(x)=\begin{cases}\frac{\tan kx}{x} & \text{for } x<0\\ 3x+2k^2 & \text{for } x\geq 0\end{cases}$. If f(x) is continuous at x = 0 then the number of values of *k* is
(A) more than 2 (B) 1
(C) 2 (D) none

60. If $f(x)=\lim\limits_{n\to\infty} x\tan^{-1}(nx)$; then f(x) is
(A) continuous at $x\in I$
(B) discontinuous at $x\in I$
(C) continuous at all $x\in R$
(D) none of these

61. If $f(x)=3+\left(1+7^{\frac{1}{1-x}}\right)^{-1}$ then
(A) $\lim\limits_{x\to1^-}f(x)=4$
(B) $\lim\limits_{x\to1^+}f(x)=3$
(C) $\lim\limits_{x\to1^+}f(x)=5$
(D) f has irremovable discontinuity at x = 1

62. Let $f(x)=\begin{cases}0, & x<1\\ 2x-2, & x\ge 1\end{cases}$. The number of solutions of the equation $f(f(f(f(x))))=x$ is

(A) 2 (B) 4
(C) 5 (D) None

63. The function $\dfrac{(3^x-1)^2}{\sin x \cdot \ell n\,(1+x)}$ is not defined for $x=0$ for the function to be continuous at the point $x=0$, $f(0)$ must have the value :

(A) e^3 (B) 1
(C) $(ln 3)^2$ (D) none

64. The value of $f(0)$ so that the function

$$f(x)=\frac{\sqrt{4-2x+x^2}-\sqrt{4+2x+x^2}}{\sqrt{2+x}-\sqrt{2-x}}$$

is continuous at $x=0$ is :

(A) 2 (B) 1
(C) $-\sqrt{2}$ (D) $-2\sqrt{2}$

65. If $f(x)=\dfrac{\log_{\sin|x|}\cos^3 x}{\log_{\sin|3x|}\cos\dfrac{x}{2}}$, $|x|<\dfrac{\pi}{3}$, $x\ne 0 = 4$, $x=0$ then the number of points of discontinuity is

(A) 0 (B) 1
(C) 2 (D) 4

66. Which one of the following functions defined below are discontinuous at the origin?

(A) $f(x)=\begin{cases}x\cos\frac{1}{x} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$

(B) $g(x)=\begin{cases}\dfrac{x^8+x^4+2x^2}{\sin x} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$

(C) $h(x)=\begin{cases}\sin x\cdot\cos\frac{1}{x} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$

(D) $\ell(x)=\begin{cases}\dfrac{x^4+x^8+2x}{\tan x} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$

67. If $f(x)=\begin{cases}2x-1, & -2\le x<0\\ x+2, & 0\le x\le 2\end{cases}$ and

$g(x)=\begin{cases}[x], & -4\le x<-2\\ x+2, & -2\le x\le 4\end{cases}$ then

(A) $\lim\limits_{x\to -2+} f\{g(x)\}=2$

(B) $f(g(x))$ is discontinuous at $x=-2$

(C) $f(g(x))$ is not defined at $x=2$

(D) None of these

68. The number of points of discontinuity of $f(x)=[4x]+\{3x\}$ in $x\in[0,5]$ is

(A) 20 points (B) 25 points
(C) 30 points (D) 35 points

69. Let $f(x)=[\tan^2 x][\cot^2 x]$ where [.] denotes greatest integer function then number of points at which function f(x) is discontinuous in $(0, 2\pi)$

(A) 0 (B) 3
(C) 4 (D) 7

70. h(x) is not a constant function and $\lim\limits_{x\to c}[h(x)]$ exists finitely for all values of $c\in R$ (where [.] denotes greatest integer function), then which of the following statement is true ?

(A) $\lim\limits_{x\to c}[h(x)]=[\lim\limits_{x\to c}h(x)],\ \forall\, c\in R$

(B) h(x) can not take any integral value

(C) h(x) can take maximum two integral values

(D) None of these

MULTIPLE CORRECT ANSWER TYPE

71. If $f(x)=[x^2]+[x]^2$, then (where [.] denotes greatest integer function

(A) f(x) is discontinuous at $x=\sqrt{2}$

(B) f(x) is continuous at $x=5^{1/4}$

(C) f(x) is continuous at $x=3^{1/5}$

(D) f(x) is discontinuous at $x=0$

72. Given the function $f(x)=\dfrac{1}{(1-x)}$, the points of discontinuity of the composite function $y=f^{3n}(x)$, where $f^n(x)=fof\ldots of$ (n times) are

(A) 0 (B) 1
(C) 3n (D) 2

73. Let $f(x)=\begin{cases}x\left[\dfrac{1}{x}\right]+x[x] & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ where [x] denotes the greatest integer function, then the correct statements are

(A) Limit exists for $x=-1$.

(B) f(x) has a removable discontinuity at $x=1$.

(C) f(x) has a non removable discontinuity at $x=2$.

(D) f(x) is discontinuous at all positive integers.

74. Let $f(x) = \lim_{n\to\infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$. If f (x) is continuous for all $x \in R$ then
(A) $a = 0$ (B) $b = 0$
(C) $a = 1$ (D) $b = 1$

75. Which one the following function(s) is/are continuous $\forall\ x \in R$
(A) $\sqrt{2\sin x} + 3$ (B) $\frac{e^x + 1}{e^x + 3}$
(C) $\left(\frac{2^{2x} + 1}{2^{3x} + 5}\right)^{5/7}$ (D) $\sqrt{\operatorname{sgn} x + 1}$

76. Which the following equations have roots ?
(A) $\cos x - x + 1 = 0$
(B) $x^5 - 18x + 2 = 0,\ x \in [-1, 1]$
(C) $x2^x = 1,\ x \in (0, 1)$
(D) $x^3 - 3x + 1 = 0,\ x \in [1, 2]$

77. $f(x) = x^4 - 14x^3 + px^2 + qx - 105$
$g(x) = x^4 + ax^3 + bx^2 + cx + 105$
the smallest root of f(x) = 0 is α and remaining root are in A.P. If the smallest root is increased by 2 then equation becomes g(x) = 0 function $F(x) = \frac{f(x)}{g(x)}$ then
(A) Domain of F(x) is $R - \{1, 3, 5, 7\}$ and range is $R - \{1, 2, \frac{3}{2}, \frac{3}{4}\}$
(B) Function F(x) is bijective
(C) Function F(x) has removable discontinuities at $x = 3, 5, 7$
(D) Function F(x) has irremovable discontinuity at $x = 1$

78. Let f(x) and g(x) be defined by f(x) = [x] and $g(x) = \begin{cases} 0 & , \quad x \in I \\ x^2 & , \quad \text{other wise} \end{cases}$, where [.] denotes greatest integer function. Then
(A) gof is continuous for all $x \in R$
(B) $\lim_{x\to 2} . fog(x) = 3$
(C) fog is continuous for all $x \in R$
(D) $\lim_{x\to 5} . fog(x) = 5$

79. If $f(x) = \cos\left[\frac{\pi}{x}\right] \cos\left(\frac{\pi}{2}(x-1)\right)$ where [x] is the greatest integer function of x, then f(x) is continuous at
(A) $x = 0$ (B) $x = 1$
(C) $x = 2$ (D) none of these

80. Which of the following statement(s) is/are correct?
(A) Let f and g be defined on R and c be any real number. If $\lim_{x\to c} f(x) = b$ and g (x) is continuous at x = b then $\lim_{x\to c} g(f(x)) = g(b)$.
(B) There exist a function $f : [0, 1] \to R$ which is discontinuous at every point in [0, 1] and |f(x)| is continuous at every point in [0, 1].
(C) If f (x) and g (x) are two continuous function defined from $R \to R$ such that f (r) = g (r) for all rational numbers 'r' then $f(x) = g(x)\ \forall\ x \in R$.
(D) If f (a) and f (b) possesses opposite signs then there must exist atleast one solution of the equation f (x) = 0 in (a, b) provided f is continuous in [a, b].

81. If $f(x) = \begin{cases} \frac{\sin ax}{bx} & , \quad x < 0 \\ ax + 1 & , \quad 0 \le x < 1 \\ cx^2 - 2 & , \quad 1 \le x < 2 \\ \frac{d(x^2 - 4)}{\sqrt{x}} & , \quad 2 \le x < 4 \\ 12 & , \quad x \ge 4 \end{cases}$
is continuous $\forall\ x \in R$ then which of the following hold good?
(A) $d = 4c$ (B) $a \ne b$
(C) $a + b + d = -3$ (D) $a + b + c + d = -\frac{5}{2}$

82. The function defined as
$$f(x) = \lim_{n\to\infty} \begin{bmatrix} \cos^{2n} x & \text{if } x < 0 \\ \sqrt[n]{1 + x^n} & \text{if } 0 \le x \le 1 \\ \frac{1}{1 + x^n} & \text{if } x > 1 \end{bmatrix}.$$
Which of the following does not hold good?
(A) continuous at x = 0 but discontinuous at x = 1
(B) continuous at x = 1 but discontinuous at x = 0
(C) continuous both at x = 1 and x = 0
(D) discontinuous both at x = 1 and x = 0

83. Which of the following functions defined below are continuous for every $x \in R$?
(A) $f(x) = \begin{bmatrix} \frac{\sin x}{x} & \text{if } x < 0 \\ x + 1 & \text{if } x \ge 0 \end{bmatrix}$

(B) $g(x)=\begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$

(C) $h(x)=\sin|x|$

(D) $k(x)=|1-x+|x||$

84. Which of the following functions is/are continuous $\forall\, x \in R$?

(A) $f(x)=\cos(x^2-2)$

(B) $f(x)=\dfrac{x}{1-\sin^2 x}$

(C) $f(x)=\begin{cases} \dfrac{\sin^2 x}{x}, & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$

(D) $f(x)=\begin{cases} \dfrac{\sin 2x}{\sin 4x}, & \text{if } x \neq 0 \\ 2 & \text{if } x=0 \end{cases}$

85. Which of the statement(s) is/are incorrect?

(A) If $f+g$ is continuous at $x=a$, then f and g are continuous at x = a.

(B) If $\lim\limits_{x\to a}(f\,g)$ exists, then $\lim\limits_{x\to a} f$ and $\lim\limits_{x\to a} g$ both exists.

(C) Discontinuity at $x=a \Rightarrow$ non existence of limit

(D) All functions defined on a closed interval attain a maximum or a minimum value on that interval.

86. Let $f(x)=\begin{cases} \left|\tan^{-1}(1/x)\right| & \text{if } x \neq 0 \\ 1 & \text{if } x=0 \end{cases}$

then which of the following do/does not hold good?

(A) f is continuous on $(-\infty, 0) \cup (0, \infty)$.

(B) f has a non removable discontinuity of finite type at $x=0$.

(C) f has a non removable discontinuity of oscillatory type at $x=0$.

(D) f has a non removable discontinuity of infinite type at $x=0$.

87. Let f(x) and g(x) be defined by f(x) = [x] and $g(x)=\begin{cases} 0 & , \quad x \in I \\ x^2 & , \quad \text{other wise} \end{cases}$, where [.] denotes greatest integer function. Then

(A) gof is continuous for all $x \in R$

(B) $\lim\limits_{x\to 2} . \text{fog}(x)=3$

(C) fog is continuous for all $x \in R$

(D) $\lim\limits_{x\to 5} . \text{fog}(x)=5$

88. The function

$f(x)=\max(\{x\},\{-x\}), x \in (-\infty,\infty)$ is

(A) continuous every where

(B) continuous at x = 0

(C) continuous for all x = n + h where $n \in I$, h = 1/2

(D) continuous every where except all $x \in I$

89. f(x) is continuous at x = 0, then which of the following are always true ?

(A) $\lim\limits_{x\to 0} f(x)=0$

(B) f (x) is non continuous at x = 1

(C) $g(x)=x^2 f(x)$ is continuous at $x=0$

(D) $\lim\limits_{x\to 0^+}(f(x)-f(0))=0$

90. The function $f(x)=\begin{cases} \dfrac{e^{[x]}-e^{[x]}}{e^x} & , \quad x<0 \\ \dfrac{\sin\{x\}}{\{\tan x\}} & , \quad x>0 \\ 2 & , \quad x=0 \end{cases}$,

where [.] and { . } represent greatest integer and fractional part functions respectively, is

(A) continuous at x = 0

(B) discontinuous at x = 0

(C) continuous at $x=\dfrac{\pi}{6}$

(D) discontinuous at $x=\dfrac{\pi}{4}$

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

91. Let $g(x)=\begin{cases} <1/x>^* & x \neq 0 \\ 0 & x=0 \end{cases}$, where $<r>^*$ is the distance from x to the integer nearest to x then

Assertion (A) : g is discontinuous at x = 0

Reason (R) : Let $x_n = 1/n$ and $x'_n=\dfrac{2}{2n+1}$;

$g(x_n)=<n>^*=0,\ g(x'_n)=\left\langle\dfrac{2n+1}{2}\right\rangle^*$

$=<n+1/2>^*=1/2$

92. Consider the function $f(x)=\begin{bmatrix} x\{x\}+1 & 0\le x<1 \\ 2-\{x\} & 1\le x\le 2\end{bmatrix}$
where {x} denotes the fractional part function.
Assertion (A) : f(x) is continuous in [0, 2]
Reason (R) : $\lim_{x\to 1} f(x)$ exists

93. **Assertion (A) :** The function $|\ln x|$ and $\ln x$ are both continuous for all $x > 0$.
Reason (R) : Continuity of $|f(x)| \Rightarrow$ Continuity of $f(x)$.

94. **Assertion (A) :** If f(x) is a continuous function such that $f(0)=1$ and $f(x)\neq x, \forall x\in R$, then $f(f(x))>x$.
Reason (R) : If $f: R\to R$, f(x) is an onto function then $f(x) = 0$ has atleast one solution.

95. Let $f(x)=\cos\left(x\cos\frac{1}{x}\right)$
Assertion (A) : $f(x)$ is discontinuous at $x = 0$.
Reason (R) : $\lim_{x\to 0^+} f(x)$ does not exist.

96. **Assertion (A) :** Let $f: [0, 1] \to R$ be a continuous function. There cannot exist such a function f which crosses the x axis infinitely often.
Reason (R) : The function $f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0\end{cases}$
crosses the x-axis infinitely often.

97. **Assertion (A) :** Let $f(x)=x^2+x^4+x^6+x^8+.....$, for all real x such that the sum converges. The number of fixed points of the function is two.
Reason (R) : Since $f(x)=\frac{x^2}{1-x^2}$, $f(x)=x$
$\Rightarrow x^2+x-1=0$. This gives two values of x.

98. Consider the functions
$f(x) = \text{sgn}(x-1)$ and $g(x)=\cot^{-1}[x-1]$
where [.] denotes the greatest integer function.
Assertion (A) : The function $F(x) = f(x) . g(x)$ is discontinuous at $x = 1$.
Reason (R) : If f(x) is discontinuous at x = a and g(x) is also discontinuous at x = a then the product function f(x) . g(x) is discontinuous at x = a.

99. Let $f(x)=\text{sgn}\,x$ and $g(x)=\begin{cases} x-1 & 0<x\le 2 \\ 1+x^2 & 2<x\le 4\end{cases}$, then
Assertion (A) : The function (fog)(x) is continuous at $x=2$.
Reason (R) : If (fog) (x) is continuous at x = a then g (x) is continuous at x = a and f (x) is continuous at $x = g(a)$.

100. **Assertion (A) :** $f(x)=[1+\cos x]$ where [x] denotes greatest integer function is discontinuous at $x=\pi$
Reason (R) : $f(x)=[x]$ where [x] denotes greatest integer function is discontinuous at all integers.

Comprehension - 1

If $f(x)=\text{maximum}\left(\cos x,\frac{1}{2},\{\sin x\}\right), 0\le x\le 2\pi$, where { . } represents the fractional part function, then

101. The number of points where f(x) is equal to 1/2 is
(A) 1 (B) 2
(C) 4 (D) infinite

102. The number of points of discontinuity of f(x) is
(A) 1 (B) 2
(C) 3 (D) 4

103. The number of points where f(x) has non-removable discontinuity is
(A) 0 (B) 1
(C) 2 (D) 3

Comprehension - 2

A strictly monotonic polynomial function $f:(0,\infty)\to(0,\infty)$ is such that $f\left(\frac{x^2}{f(x)}\right)\equiv x$.

104. If $f(1) = 2$ then the value of f(2) is
(A) 8 (B) 4
(C) 16 (D) None

105. The number of fixed points of f (i.e. number of solution of $f(x) = x$) is
(A) 3 (B) 2
(C) 1 (D) None

106. If $\lim_{x\to 1}\frac{\ln f(x)-\ln f(1)}{\sin^n \pi x}$ exists with nonzero value, then the value of n is
(A) 1 (B) 2
(C) 3 (D) None

Comprehension - 3

Define $f: R\to R$ by $f(x) = x^2 + 1$ for $-1 < x \le 1$ and $f(x+2)=\frac{1}{f(x)}$ for all $x\in R$.

107. The fundamental period of the function f(x) is

(A) 2 (B) 4
(C) 8 (D) None

108. $\lim_{x \to 1^+} f(x)$ is equal to

(A) $\frac{1}{3}$ (B) $\frac{3}{4}$
(C) $\frac{1}{2}$ (D) $\frac{5}{12}$

109. The number of points of discontinuity in the function y = f(x) over the interval [0, 4] is :

(A) 0 (B) 1
(C) 2 (D) 3

Comprehension - 4

Consider $a \in R^+$, $f(x) = |x-a|$

$$g(x) = \begin{cases} f(x)\sec\left(\frac{\pi x}{2a}\right) & \text{for } x < a \\ -\cot\left(\frac{\pi x}{2a}\right)\operatorname{cosec}(f(x)) & \text{for } x > a \end{cases}$$

$$h(x) = \lim_{n\to\infty}\left(2^n \sin\left(\frac{x}{2^n}\right)\right)$$

110. The value of g(a) so that g is continuous at x = a

(A) is $-\frac{2a}{\pi}$ (B) is $\frac{a}{2\pi}$
(C) is 1 (D) can not be determined

111. The number of possible ordered pairs (a, g(a)) is

(A) 0 (B) 1
(C) 2 (D) more than 2

112. If $\lim_{x\to a}(\cos(x-a))^{\frac{6}{(x-a)\sin(2x-2a)}} = e^{-kh([a])}$ then k equals
(where [.] denotes greatest integer function)

(A) 0 (B) 2/3
(C) 3/2 (D) 6

Comprehension - 5

Let the Heaviside step function be defined as

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

113. The solutions of the equation $x^2 - x + H(x) - 2 = 0$ are

(A) $\{-1, 2\}$ (B) $\{-\sqrt{2}, -1, 2\}$
(C) $\{\pm\sqrt{2}, -1, 2\}$ (D) $\{-\sqrt{2}, 2\}$

114. The solution set of the inequality $H(1-x^2) > |\sin \pi/2\, x|$ is

(A) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (B) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(C) $R - \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (D) None of these

115. The number of points of discontinuity in the function $y = H(\cos 2x), x \in [0, 2\pi]$ is

(A) 4 (B) 5
(C) 6 (D) None

Match the Columns Exercise for JEE Advanced

116. **Column - I** / **Column - II**

(A) If $f(x) = \begin{cases} \frac{1+3\cos x}{x^2}, & x < 0 \\ b\tan\left(\frac{\pi}{[x+3]}\right), & x \geq 0 \end{cases}$ is continuous at x = 0, then (where [.] denotes the greatest integer function) — (P) $|a+b| = 0$

(B) If $f(x) = \begin{cases} -2\sin x, -\pi \leq x \leq -\frac{\pi}{2} \\ a\sin x + b, -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x, \frac{\pi}{2} \leq x \leq \pi \end{cases}$ is continuous in $[-\pi, \pi]$, then — (Q) $|a-b| = 2$

(C) If $f(x)=\begin{cases} \left(\dfrac{3}{2}\right)^{(\cos 3x)/(\cot 2x)} & , \quad 0<x<\dfrac{\pi}{2} \\ b+3 & , \quad x=\dfrac{\pi}{2} \\ (1+|\cos x|)^{\left(\frac{a|\tan x|}{b}\right)} & , \quad \dfrac{\pi}{2}<x<\pi \end{cases}$ is continuous at $x=\dfrac{\pi}{2}$, then (R) $[a-2b]=-2$

(D) If $f(x)=\begin{cases} a+\dfrac{\sin[x]}{x}, & x>0 \\ 2, & x=0 \\ b+\left[\dfrac{\sin x - x}{x^3}\right], & x<0 \end{cases}$ (S) $|a+2b|=4$

(where [.] denotes the greatest integer function), is continuous at x = 0, then b is equal to (T) $|a-b|=1$

117.

	Column – I		Column – II
(A)	Given $f(u)=\dfrac{1}{u^2+u-2}$, where $u=\dfrac{1}{x-1}$, then f(x) is	(P)	continuous at x = 0
(B)	If $f(x)=\text{sgn}\, x\,(1-x^2)$, then f(x) is	(Q)	discontinuous at x = 1, 1/2, 2
(C)	If $\lim\limits_{x\to 0}\dfrac{f(x)}{x}$ exists and f(0) = 0, then f(x) is	(R)	discontinuous function
(D)	The function $f(x)=\lim\limits_{n\to\infty}\dfrac{nx+\text{sgn}\, x}{1+n}$ is	(S)	discontinuous at x = 2

119.

	Column - I		Column - II
(A)	The number of natural numbers less than the fundamental period of $\sin^2 x+\sec^2 x-\tan^2 x$ is	(P)	1
(B)	The number of points of discontinuity of the function $f(x)=[x]+\{2x\}+[3x]$ for $x\in[0,1]$, where [.] and { . } represent greatest integer and fractional part functions is	(Q)	2
(C)	$\left[\lim\limits_{x\to 0}\dfrac{\sin x(1+\cos x)}{x\cos x}\right]$, where [.] represents greatest integer function, is equal to	(R)	3
(D)	The number of solutions of the equation $\sin^{-1}x-2\cos^{-1}(1+x)=0$ is	(S)	4

120.

	Column-I		Column-II
(A)	In a ΔABC maximum value of $\cos^2 A+\cos^2 B+\cos^2 C$, is	(P)	3/4
(B)	If a, b are c are positive and 9a + 3b + c = 90 then the maximum value of (log a + log b + log c) is (base of the logarithm is 10)	(Q)	2

(C) $\lim_{x\to 0} \dfrac{\tan x\sqrt{\tan x} - \sin x\sqrt{\sin x}}{x^3 \cdot \sqrt{x}}$ equals | (R) 3

(D) If $f(x) = \cos(x \cos \frac{1}{x})$ and $g(x) = \dfrac{\ln(\sec^2 x)}{x \sin x}$ are both continuous at x = 0 then f (0) + g (0) equals | (S) non existent

121. **Column I** | **Column II**

(A) If $f(x) = \lim_{n\to\infty} \dfrac{1}{1 + n\sin^2 \pi x}$, then | (P) f(x) is continuous $\forall\, x \in R$

(B) If $f(x) = \lim_{n\to\infty} \dfrac{x}{1 + (4\sin^2 x)^n}$, then | (Q) f(x) is discontinuous at x = 1

(C) If $h(x) = \lim_{n\to\infty}\left(\dfrac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}\right)$ is continuous $\forall\, x \in R$, and $f(x) = [\tan(a + b - 1)x]$, then | (R) f(x) is discontinuous at $x = 2n\pi + \dfrac{\pi}{6}$

(D) If $f(x + y) = f(x).\, f(y), \forall x, y \in R$ and $f(x) = 1 + g(x).G(x)$; $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} G(x)$ is finite real number, then | (S) f(x) is continuous at $x = \dfrac{5\pi}{6}$

Review Exercises for JEE Advanced

1. Test the continuity of the function
$f(x) = \left|\left(x + \frac{1}{2}\right)[x]\right|$, in the interval $-2 \le x \le 2$. Also draw the graph of $y = f(x)$, where [.] denotes greatest integer function.

2. Discuss the continuity of the function $f(x) = [[x]] - [x - 1]$, where [·] denotes the greatest integer function.

3. Let f(x) be defined in the interval [–2, 2] such that
$f(x) = 1, -2 \le x \le 0$
$= x - 1, 0 < x \le 2$
and $g(x) = f(|x|) + |f(x)|$. Test the continuity of g(x) in [–2, 2].

4. If f(x) be defined as

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x > 0 \\ 1 - x\cos x, & x \le 0 \end{cases}$$

then discuss the continuity $y = [f(x)]$ in $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$ where [.] denotes G.I.F.

5. Let $f(x) = \dfrac{x^2}{2} + 1.\ 0 \le x < 1$
$= 2x^2 - 3x + \dfrac{3}{2}, 1 \le x \le 2.$
Discuss the continuity of $g(x) = f(x) + f(x - 1)$.

6. Discuss the continuity of the function
$$f(x) = \lim_{n\to\infty} \frac{(1 + \cos x)^n + 5\lambda nx}{2 + (1 + \cos x)^n}$$

7. Let $f_n(x) = \cos^n x$ and $g(x) = \lim_{x\to\infty} \sum_{k=0}^{n} f_k\left(\dfrac{x}{4}\right)$. If g(x) is continuous is (0, c), then find the largest value of c.

8. Examine the function f defined on R by setting
$f(x) = \dfrac{e^{1/x}\sin(1/x)}{1 + e^{1/x}}$, when $x \ne 0$, $f(0) = 0$
for points of discontinuity, if any.

9. Examine each of the following functions for points of discontinuity and the nature of discontinuity :
(i) $f(x) = (x - [x])^2$, for all $x \ge 0$
(ii) $f(x) = [x] + (x - [x])^2$, for all $x \ge 0$

10. If the function

$f(x) = \left[\frac{(x-2)^3}{a}\right]\sin(x-2) + a\cos(x-2)$ where [.] denotes the greater integer function, is continuous in [4, 6] , then find the values of a.

11. Find the value of a & b for which

$$f(x) = \begin{cases} (\sin x + \cos x)^{\text{cosec } x} , & -1/2 \le x < 0 \\ a , & x = 0 \\ \dfrac{e^{1/x} + e^{2/x} + e^{3/|x|}}{ae^{3/x} + be^{3/|x|}} , & 0 < x \le 1/2 \end{cases}$$

is continuous at x = 0.

12. Prove that the function $f(x) = \begin{cases} 2x, & -1 \le x \le 0 \\ \dfrac{2x+1}{2}, & 0 < x \le 1 \end{cases}$ is discontinuous at x = 0 but still has both maximum and minimum values on [–1, 1].

13. Let $f(x) = \begin{cases} x^n \dfrac{e^{1/x}}{1+e^{1/x}} , & x < 0 \\ 0 , & x = 0 \\ x^n \sin\dfrac{1}{x} , & x > 0 \end{cases}$. Find the smallest $n \in W$ such that f(x) is continuous.

14. If $f(x) = \begin{cases} x^2 + ax + 1 , & x \in Q \\ ax^2 + 2x + b, & x \notin Q \end{cases}$ is continuous at x = 1 & e then a & b.

15. Let $f(x) = \frac{x^2}{e^x}$,$g(x) = \frac{2\ln x}{x}$. Prove that there exist a point ‘c’ between 1 and e such that f(c) = g(c).

16. Prove that, if f(x) is continuous on (a, b) and $x_1, x_2, \ldots\ldots\ldots x_n$ are some values of x from this interval, then we can find x = c, c ∈ (a, b) such that $f(c) = (1/n)\,[f(x_1) + f(x_2) + \ldots\ldots\ldots\ldots + f(x_n)]$.

17. Let $f(x) = \begin{cases} |x+1| , & x \le 0 \\ x , & x > 0 \end{cases}$;

$g(x) = \begin{cases} |x| + 1 , & x \le 1 \\ -|x-2| , & x > 1 \end{cases}$.

Discuss the continuity of f + g.

18. Let $f(x) = \begin{cases} 2x - 1, & -2 \le x < 0 \\ x + 2, & 0 \le x \le 2 \end{cases}$ and

$g(x) = \begin{cases} [x], & -4 \le x < -2 \\ x + 2, & -2 \le x \le 4 \end{cases}$

Discuss the continuity of fog(x) over its domain.

19. Discuss the continuity of $f : R^+ \to R$ defined as :

f(x) = x when x is irrational, $f(x) = \left(\frac{1+p^2}{1+q^2}\right)^{1/2}$ when x is rational number of the form p/q.

20. Use the Intermediate Value Theorem to show that there is a right circular cylinder of height h and radius less than r whose volume is equal to that of a right circular cone of height h and radius r.

21. Use Intermediate Value Theorem to locate all discontinuities of the function $f(x) = \frac{x}{x^3 - 3x + 1}$.

22. Prove that if a and b are positive, then the equation $\frac{a}{x-1} + \frac{b}{x-3} = 0$ has atleast one solution in the interval (1, 3).

23. Let $f(x) = x(1-x^2), x \in R$ and

$g(x) = \begin{cases} \dfrac{|x|}{x} , & x \ne 0 \\ 0 , & x = 0 \end{cases}$

Discuss the continuity of (fog) x and (gof) x.

24. Show that any continuous function defined for all real x and satisfying the equation f(x) = f(2x + 1) for all x must be a constant function.

25. Let f(x) be a continuous function in [–1, 1] and satisfies $f(2x^2 - 1) = 2x\, f(x)\ \forall\ x \in [-1, 1]$. Prove that f(x) is identically zero ∀ x ∈ [–1, 1].

26. If $g(x) = \begin{cases} [f(x)] , & x \in (0, \pi/2) \cup (\pi/2, \pi) \\ 3 , & x = \pi/2 \end{cases}$ and

$f(x) = \frac{2(\sin x - \sin^n x) + |\sin x - \sin^n x|}{2(\sin x - \sin^n x) - |\sin x - \sin^n x|}, n \in R$

where [.] denotes the greatest integer function. Prove that g(x) is continuous at x = π/2 when n > 1.

27. Prove that the function $f(x) = (-1)^{[x^3]}$, where [.] denotes the greatest integer function, is discontinuous for $x = n^{1/3}$, n ∈ I.

28. Let f be a continuous function in [0, 4] and f(0) = f(4). Prove that these exists point x = c ∈ [0, 2] such that f(c) = f(c + 2).

29. Let f be a function such that $f(xy) = f(x)\,f(y^3)$ for all x and y. If f(x) is continuous at x = 1, show that f(x) is continuous at all $x \neq 0$.

30. If $f(x) = \begin{cases} |x+1|; & x \le 0 \\ x; & x > 0 \end{cases}$ and

$g(x) = \begin{cases} |x|+1; & x \le 1 \\ -|x-2|; & x > 1 \end{cases}$

Draw the graph of f + g and discuss its continuity.

31. Let f(x) = x, if x is not the reciprocal of a positive integer; but $f(x) = x^2$ if x = 1/n for some positive integer n. Show that f is continuous at $x_0 = 0$. Is f discontinuous at any point in R?

32. Consider the function $f : R \to R$ defined by

$$f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1 + x^{2n}}$$

in the interval $[0, \pi/2]$ and explain why the function does not vanish anywhere in this interval, although f(0) and $f\left(\frac{1}{2}\pi\right)$ differ in sign.

33. A continuous function $f : R \to R$ satisfies $f(x + f(x)) = f(x)\ \forall\, x \in [0, 1]$. Prove that f is constant.

34. Let f be a non-zero function whose domain is the set of all real numbers satisfying $f(x + h) = A\,f(x)\,f(h)$ $(A \neq 0)$. If $f(0) \neq 0$, and f is continuous at x = 0 then show that f is a continuous function.

35. Let f be a continuous function on R. If $f(1/3^n) = (\cos e^n)\, 3^{-n^2} + \dfrac{n^2}{n^2+n+1}$ then find f(0).

Target Exercises for JEE Advanced

1. Let $f(x) = \begin{cases} 1, & |x| \ge 1 \\ \dfrac{1}{n^2}, & \dfrac{1}{n} \le |x| < \dfrac{1}{n-1} \\ 0, & x = 0 \end{cases}$, n = 2, 3,..... prove that f(x) is discontinuous at infinitely many points.

2. Let $f(x) = \begin{cases} \dfrac{(1-x)^{-1/x} - (1+x)^{1/x}}{x}, & -1 < x < 0 \\ e, & x = 0 \\ (\log_{\sin x} \sin 2x)^{\ln \sin 2x}, & 0 < x < 1 \end{cases}$

Examine the continuity of f(x) at x = 0.

3. Let $f(x) = x^3 - 9x^2 + 15x + 7$, and

$g(x) = \begin{cases} (\min f(t) : 0 \le t \le x), & 0 \le x \le 6 \\ x - 24, & x > 6 \end{cases}$

Draw the graph of g(x) and discuss the continuity of g(x).

4. Discuss the continuity of the function

$f(x) = \begin{cases} \dfrac{2(x-x^3) + |x-x^3|}{2(x-x^3) - |x-x^3|}, & x \neq 0 \\ \dfrac{1}{3}, & x = 0 \\ 3, & x = 1 \end{cases}$ at x = 0,1.

5. Let f(x) = x when x is rational
= 1–x when x is irrational.

Show that f(x) assumes every value between 0 and 1 once and once only as x increases from 0 to 1, but is discontinuous for every value of x except $x = \frac{1}{2}$.

6. Let f be a continuous functions on [–1, 1] such that $(f(x))^2 + x^2 = 1$, for all, $x \in [-1, 1]$. Show that either $f(x) = \sqrt{1-x^2}$ for all x in [–1, 1] or $f(x) = -\sqrt{1-x^2}$ for all x in [–1, 1].

7. Let $y_n(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + ... + \dfrac{x^2}{(1+x^2)^{n-1}}$ and $y(x) = \lim_{n\to\infty} y_n(x)$

Discuss the continuity of $y_n(x)$(n = 1, 2, 3,.....n) and y(x) at x = 0

8 Show that the function

$f(x) = \frac{1}{2} - x + \frac{1}{2}[2x] - \frac{1}{2}[1 - 2x]$ assumes every value between 0 and 1 once and once only as x increases from 0 to 1, but is discontinuous for x = 0, $x = \frac{1}{2}$ and x = 1.

9. Let I be a closed and bounded interval on the line and let f be continuous on I. Suppose that for each $x \in I$, there exists a $y \in I$ such that

$|f(y)| \le \frac{1}{2}|f(x)|$.

Prove the existence of a $t \in I$ such that f(t) = 0.

10. Let $f : R \to R$ be a continuous function such that $\lim_{x\to-\infty} f(x) = 0 = \lim_{x\to\infty} f(x)$. Prove that if f is strictly negative somewhere on R then f attains a finite absolute minimum on R, and that if f is strictly positive somewhere on R then f attains a finite absolute maximum on R.

11. Let $f : [0, 1] \to [0, 2]$ be continuous. Show that there exists a point $x \in [0, 1]$ such that $f(x) = 2x$.

12. Let $f : [0, 1] \to R$ be a continuous function such that $f(0) = f(1)$, then prove that there is a solution of the equation $f(x) - f\left(x + \frac{1}{n}\right) = 0$, in $\left[0, \frac{n-1}{n}\right]$ for every natural number n .

13. How many continuous functions are there on R which satisfy $(f(x))^2 = x^2$ for all $x \in R$?

14. Let f be continuous on [a,b], let f(x) = 0 for exactly one c in [a, b] and let f(x) > 0 for some x such that $c < x \le b$ and let f(x) < 0 for some x such that $a \le x < c$. What can we say about f for all x in [a,b].

15. Let f be a function that satisfies the conclusion of the intermediate value theorem on a closed interval I and let f be injective on I then prove that f must be continuous on I.

16. A function f is said to satisfy a Lipschitz condition on a given interval if there is a positive constant M such that $|f(x) - f(y)| < M\,|x - y|$.
for all x and y in the interval (with $x \ne y$). Suppose f satisfied a Lipschitz condition on an interval and let c be a fixed number chosen arbitrarily from the interval. Use limit to prove that f is continuous at c.

17. Find the points of discontinuity of the function $f : (0, \infty) \to R$ where

$$f(x) = \begin{cases} \dfrac{1}{p+q} & \text{if } x \in Q \cap (0,\infty), x = \dfrac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \in (0,\infty) - Q \end{cases}$$

18. Let f, g be continuous function from [0, 1] to [0, 1] such that
$f(g(x)) = g(f(x)) \ \forall x \in [0, 1]$.
Prove that f and g have a common fixed point in [0, 1].

19. Let f be a function such that $|f(u) - f(v)| \le |u - v|$ for all u and v in an interval [a, b].

(i) Prove that f is continuous at each point of [a, b].

(ii) Assume that f is integrable on [a, b] prove that $\left|\int_a^b f(x)dx - (b-a)f(a)\right| \le \frac{(b-a)^2}{2}$

(iii) More generally, prove that for any c in [a, b], we have $\left|\int_a^b f(x)dx - (b-a)f(c)\right| \le \frac{(b-a)^2}{2}$

20. Let f be continuous and strictly monotonic on the positive real axis and let g denote the inverse of f. If $a_1 < a_2 < ... < a_n$ are n given positive real numbers, we define their mean value (or average) with respect to a f to be the number M_f defined as follows:

$$M_f = g\left(\frac{1}{n}\sum_{i=1}^{n} f(a_i)\right)$$

Prove that (i) $f(M_f) = \frac{1}{n}\sum_{i=1}^{n} f(a_i)$ (ii) $a_1 < M_f < a_n$

If $h(x) = af(x) + b, a \ne 0$, then show that $M_h = M_f$.

21. If f is a function, then by a chord of f we shall mean a line segment whose ends are on the graph of f. Now let f be continuous throughout [0,1] and let f(0) = f(1) = 0.

(i) Explain why there is a horizontal chord of f of length $\frac{1}{2}$.

(ii) Explain why there is a horizontal chord of f of length 1/n, where n = 1, 2, 3, 4,....

(iii) Must there exist a horizontal chord of length $\frac{2}{3}$?

(iv) What is the answer to (iii) if it is given that $f(x) \ge 0$ for all x in [0, 1]

22. Classify the points of discontinuity of the function f(x) = pnx, where pnx denotes the positive or negative excess of x over the nearest integer; when x exceeds an integer by 1/2 let pnx = 0.

23. Discuss the continuity at x = 1 of the functions :

(i) $f(x) = \lim_{n\to\infty} \frac{x + x^n \sin x}{1 + x^n}$;

(ii) $\phi(x) = \lim_{n\to\infty} \frac{x + x^n}{1 + ax^n}$.

24. If $f(x) = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{[2rx]}{n^2}$, discuss the continuity of f(x) where [.] denotes the greatest integer function.

25. Consider the functions y(x) = sinx and $z_k(x) = ke^{-x}$. Now define a new function f by f(k) = {smallest positive solution of $y(x) = z_k(x)$} Explain why the function f(k) is not continuous on the interval $0 \le k \le 10$.

26. Let f and g be continuous on an interval I, and let $(f(x))^2 (g(x))^2 = 1$ for all $x \in I$. Prove that either f(x) = 1/g(x) for all $x \in I$ or f(x) = –1/g(x) for all $x \in I$.

27. Discuss the following function in the interval $0 < x < 1$ for continuity : let $x = 0.a_1 a_2 a_3 \ldots$ be the decimal representation of x. Let a_k be the first digit equal to 7. Put $f(x) = 0.a_1 a_2 \ldots a_k$ if a_k exists and f(x) = x if no digit equal to 7 occurs.

28. A function f is defined on [0, 1] as follows :

$f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$ $(n \in W)$, $f(0) = 0$

Show that f is discontinuous at the points $\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots$ and examine the nature of discontinuity.

29. $f(x) = \dfrac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$ for > 0

$= \dfrac{\ln(1 + x + x^2) + \ln(1 - x + x^2)}{\sec x - \cos x}$ for x < 0.

If f is continuous at x = 0, find 'a'.

Now if $g(x) = \ln\left(2 - \frac{x}{a}\right) \cdot \cot(x - a)$ for $x \ne a$, $a > 0$. If g is continuous at x = a then show that $g(e^{-1}) = -e$.

30. Given $f(x) = \sum_{r=1}^{n} \tan\left(\frac{x}{2^r}\right) \sec\left(\frac{x}{2^{r-1}}\right)$; $r, n \in N$ and

$$g(x) = \lim_{n\to\infty} \frac{\ell n\left(f(x) + \tan\frac{x}{2^n}\right) - \left(f(x) + \tan\frac{x}{2^n}\right)^n \cdot \left[\sin\left(\tan\frac{x}{2}\right)\right]}{1 + \left(f(x) + \tan\frac{x}{2^n}\right)^n}$$

$= k$ for $x = \frac{\pi}{4}$ and the domain of g(x) is $(0, \pi/2)$.

where [ ] denotes the greatest integer function. Find the value of k, if possible, so that g(x) is continuous at $x = \pi/4$. Also state the points of discontinuity of g(x) in $(0, \pi/4)$, if any.

31. Given the function $g(x) = \sqrt{6 - 2x}$ and $h(x) = 2x^2 - 3x + a$. Then

(i) evaluate h(g(2))

(ii) If $f(x) = \begin{cases} g(x), & x \le 1 \\ h(x), & x > 1 \end{cases}$, find 'a' so that f is continuous.

32. Let

$$f(x) = \begin{cases} \dfrac{\left(\frac{\pi}{2} - \sin^{-1}(1 - \{x\}^2)\right) \cdot \sin^{-1}(1 - \{x\})}{\sqrt{2}(\{x\} - \{x\}^3)} & \text{for } x \ne 0 \\ \dfrac{\pi}{2} & \text{for } x = 0 \end{cases}$$

where {x} is the fractional part of x.

Consider another function g(x); such that

$g(x) = f(x)$ for $x \ge 0$

$= 2\sqrt{2}\, f(x)$ for $x < 0$

Discuss the continuity of other functions f(x) & g(x) at x = 0.

33. Let f be the function defined on [0,1] by setting

$f(x) = 2rx$, when $\frac{1}{r+1} \le x < \frac{1}{r}$, $r = 1, 2, 3, \ldots$

$f(0) = 0$, $f(1) = 1$.

Examine for continuity the function f at the points $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{r}, \ldots$, and 0.

34. Let f be defined on R by setting

$$f(t) = \begin{cases} t, & \text{if } 0 \le t < \frac{1}{2} \\ 0, & \text{if } t = \frac{1}{2} \\ t - 1, & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

and f(n+t) = f(t), when n is any integer. Determine the points of discontinuity of f.

35. Sketch the graph of the function y = f(x), where

$$f(x) = \lim_{n\to\infty} \frac{\ln(2 + x) - x^{2n} \sin x}{1 + x^{2n}}$$

in the interval $0 \le x \le \frac{1}{2}\pi$ and explain why the function does not vanish anywhere in this interval, although f(0) and $f\left(\frac{1}{2}\pi\right)$ differ in sign.

Previous Year's Question (JEE Advanced)

A. Fill in the blanks:

1. Let $f(x)=\begin{cases}\dfrac{(x^3+x^2-16x+20)}{(x-2)^2}, & \text{if } x\neq 2\\ k, & \text{if } x=2\end{cases}$

If f(x) is continuous for all x, then k =....... **[IIT - 1981]**

2. A discontinuous function y = f(x) satisfying $x^2+y^2=4$ is given by f(x) =.......... **[IIT - 1982]**

3. Let $f(x)=[x]\sin\left(\dfrac{\pi}{[x+1]}\right)$, where [.] denotes the greatest integer function. The domain of f is .... and the points of discontinuity of f in the domain are................ **[IIT - 1996]**

4. Let f(x) be a continuous function defined for $1\le x\le 3$. If f(x) takes rational values for all x and f(2) = 10, then f(1.5) = ... **[IIT - 1997]**

B. Multiple Choice Questions with ONE correct answer:

5. The function $f(x)=\dfrac{\ln(1+ax)-\ln(1-bx)}{x}$ is not defined at x = 0. The value which should be assigned to f at x = 0 so that it is continuous at x = 0, is **[IIT - 1983]**

(A) a – b (B) a + b
(C) ln a – ln b (D) none of these

6. The function $f(x)=[x]\cos\left(\dfrac{2x-1}{2}\right)\pi$, [.] denotes the greatest integer function, is discontinuous at **[IIT - 1995]**

(A) All x (B) All integer points
(C) No x (D) x which is not an integer

7. The function $f(x)=[x]^2-[x^2]$ (where [y] is the greatest integer less than or equal to y), is discontinuous at **[IIT - 1999]**

(A) all integers
(B) all integers except 0 and 1
(C) all integers except 0
(D) all integers except 1

C. Multiple Choice Questions with ONE or MORE THAN ONE correct answer :

8. If f (x) = (x – 1)/2, then on the interval [0, π] **[IIT - 1989]**

(A) tan (f(x)) and 1/f(x) are both continuous
(B) tan (f(x)) and 1/f(x) are both discontinuous
(C) tan (f(x)) and $f^{-1}(x)$ are both continuous
(D) tan (f(x)) is continuous but 1/f(x) is not.

9. The following function are continuous on (0, π) **[IIT - 1991]**

(A) tan x

(B) $\int_0^x t\sin\dfrac{1}{t}dt$

(C) $\begin{cases}1, & 0<x\le\dfrac{3\pi}{4}\\ 2\sin\dfrac{2}{9}x, & \dfrac{3\pi}{4}<x<\pi\end{cases}$

(D) $\begin{cases}x\sin x, & 0<x\le\pi/2\\ \dfrac{\pi}{2}\sin(\pi-x); & \dfrac{\pi}{2}<x<\pi\end{cases}$

10. If $f(x)=\begin{cases}x\sin x, & \text{when } 0<x\le\pi/2\\ \dfrac{\pi}{2}\sin(\pi+x); & \text{when } \dfrac{\pi}{2}<x<\pi\end{cases}$, then **[IIT - 1991]**

(A) f(x) is discontinuous at $x=\dfrac{\pi}{2}$
(B) f(x) is continuous $x=\dfrac{\pi}{2}$
(C) f(x) is continuous at x = 0
(D) None of these

11. For every integer n, let an and bn be real numbers. Let function $f:\mathbb{R}\to\mathbb{R}$ be given by

$f(x)=\begin{cases}a_n+\sin\pi x, & \text{for } x\in[2n,2n+1]\\ b_n+\cos\pi x, & \text{for } x\in[2n-1,2n]\end{cases}$,

for all integers n,.

(A) $a_{n-1}-b_{n-1}=0$ (B) $a_n-b_n=1$
(C) $a_n-b_{n+1}=1$ (D) $a_{n-1}-b_n=-1$

[IIT - 2012]

F. Subjective Problems:

12. Let $f(x+y)=f(x)+f(y)$ for all x and y. If the function f(x) is continuous at x = 0, then show that f(x) is continuous at all x **[IIT - 1981]**

13. Determine the values a, b, c for which the function

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x}, & x<0\\ c, & x=0\\ \dfrac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}}, & x>0\end{cases}$$

is continuous at x = 0 **[IIT - 1982]**

14. Let $f(x)=\begin{cases}1+x, & 0\le x<2\\ 3-x, & 2\le x<3\end{cases}$;

Determine the form of $g(x)=f[f(x)]$ and hence find the points of discontinuity of g, If any. **[IIT - 1983]**

15. Let f(x) be a continuous and g(x) be discontinuous function , prove that f(x) + g(x) is discontinuous function. **[IIT - 1987]**

16. Find the value of a and b so that the function **[IIT - 1989]**

$$f(x)=\begin{cases}x+a\sqrt{2}\sin x, & 0\le x<\pi/4\\ 2x\cot x+b & \pi/4\le x\le\pi/2\\ a\cos 2x-b\sin x & \pi/2<x\le\pi\end{cases}$$

is continuous for $0\le x\le\pi$

17. Let $f(x)=\dfrac{1-\cos 4x}{x^2}$ $\quad x<0$ **[IIT - 1990]**

$=a \quad x=0$

$=\dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} \quad x>0$

Determine the value of a, if possible, so that the function is continuous at x = 0.

18. Let $f(x)=\begin{cases}\{1+|\sin x|\}^{a/|\sin x|} & ; & -\dfrac{\pi}{6}<x<0\\ b & ; & x=0\\ e^{\tan 2x/\tan 3x} & ; & 0<x<\dfrac{\pi}{6}\end{cases}$ **[IIT - 1994]**

Determine a and b such that f(x) is continuous at x = 0.

$\frac{5}{3}$

ANSWERS

CONCEPT PROBLEMS—A

2. 1 **3.** $-\dfrac{1}{8}$

4. a = 0 **5.** 6

6. (i) 7/3 (ii) π/4

7. continuous

8. f(0) = e

PRACTICE PROBLEMS—A

9. discontinuous **10.** $\dfrac{5}{6}$

12. 1 **13.** $a=\dfrac{\pi}{6}, b=\dfrac{-\pi}{12}$

14. $a=-\text{In } 3, b=\dfrac{1}{3}, c=1$ **16.** a + b = 0

17. $a=2/3, b=e^{2/3}$. **18.** 1

19. 1

CONCEPT PROBLEMS—B

1. The tangent is continuous everywhere except at $x=\dfrac{1}{2}\pi+n\pi$, where n is any integer; the cotangent is continuous everywhere except at x = nπ, where n is any integer.

2. (i) $x=(2n+1)\dfrac{\pi}{2}, n\in I$ (ii) ±1.

4. **5.** No

6. a = (2cosc – b)/c² if c ≠ 0; if c = 0 there is no solution unless b = 2, in which case any a will do.

7. (i) a = 16 b = 4

(ii) a = 4/3, 14/3, ... , $b=\sqrt{3}$

8. (i) $x = \pm\sqrt{n}, n \in N$, (ii) $x = n^2, n \in N$,
(iii) $x = n, n \in I$, (iv) $x = n/2, n \in I$,
(v) $x \in I$

9. (i) No (ii) No
(iii) $f(x) \to 0$ as $x \to 0$, define $f(0) = 0$ for continuity at 0

10. True

11. (i) discontinuous at 0, left continuous
(ii) discontinuous at 0, right continuous; discontinuous at 1; left continuous.

12. (i) $a + b + 1 = 0$ (ii) $a = 3/2 + 2n$.

13. (i) 0 (ii) $n\pi/2, n \in I$.

14. (i) 1 (ii) 0
(iii) yes, define $f(0) = 0$

15. R

16. (i) Discontinuities at $x = 0$ and $x = \pm\frac{2}{\pi}, \pm\frac{2}{3\pi}, \ldots, \pm\frac{2}{(2n+1)\pi}, \ldots\ n \in I$.
(ii) Discontinuities at $x = -2, -1, 0, 2$
(iii) $f(x)$ is discontinuous at $x \in I$.

PRACTICE PROBLEMS—B

17. $a = 8$

18. $\{1, \frac{3}{2}, \frac{5}{2}, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5\}$

21. $x \in [0, 1) \cup [2, \infty)$

22. 12, 4, 2

23. No

24. all $x \in R$ except $x = n\pi + \frac{\pi}{4}, n \in I$

25. (i) 3 (ii) dne
(iii) no (iv) at $x = n + 1/2, n \in I$.

26. Discontinuous at $x = n\pi \pm \frac{\pi}{3}, n \in I$.

27. Discontinuous at all integral values in [–2, 2]

28. Continuous on R.

29. $a = 0, b = 1$

30. discontinuous at $x = 0$.

CONCEPT PROBLEMS—C

1. isolated point removable discontinuity of first kind

2. irremovable finite discontinuity of first kind

4. $x = -2$ is a discontinuity of the first kind (the jump being equal to 2)

5. No, it has a discontinuity of second kind.

6. Infinite discontinuity of second kind.

7. No

8. (i) $g(x) = x - 4$,
(ii) irremovable discontinuity

PRACTICE PROBLEMS—C

9. (i) $x = 0$ is a point of removable discontinuity,
(ii) $x = 0$ is a point of discontinuity of the second kind
(iii) $x = k$ ($k \in I$) are points of discontinuity of the first kind,
(iv) at the points $x = 1$ and $x = -1$ the function is continuous and the other points are points of discontinuity of the second kind,
(v) $x = -1$ is a point of discontinuity of the second kind,
(vi) $x = 0$ is a point of discontinuity of the first kind
(vii) $x = 1$ is a point of discontinuity of the first kind
(viii) $x = -1$ and $x = 3$ are points of discontinuity of the second kind,
(ix) $x = 1$ is a point of removable discontinuity,
(x) $x = -1$ is a point of discontinuity of the first kind.

10. (i) missing point discontinuity,
(ii) isolated point discontinuity,
(iii) infinite discontinuity,
(iv) infinite discontinuity,
(v) infinite discontinuity,
(vi) infinite discontinuity,
(vii) of second kind discontinuity

11. No

13. Irremovable discontinuity

14. (i) $x = -2$ is a discontinuity of second kind; $x = 2$ is a removable discontinuity.
(ii) $x = -1$ is a removable discontinuity; $x = 1$ is a discontinuity of first kind.
(iii) $x = 1$ is a discontinuity of first kind.

15. (i) $x = 1$ is a discontinuity of first kind.
(ii) continuous.

CONCEPT PROBLEMS—D

2. (i) Yes, Hint. If the function $\varphi(x) = f(x) + g(x)$ is continuous at the point $x = x_0$, then the function $g(x) = \varphi(x) - f(x)$ is also continuous at this point;
(ii) No. Example: $f(x) = -g(x) = \text{sgn } x$; both function are discontinuous at the point $x = 0$, and their sum is identically equal to zero, and

is, hence continuous.

3. (i) No. Example f(x) = x is continuous everywhere, and g(x) = sin $\frac{\pi}{x}$ for x ≠ 0, g(0) = 0 being discontinuous at the point x = 0. The product of these function is a function continuous at x = 0 since $\lim_{x\to 0} x \sin\frac{\pi}{x} = 0$;

(ii) No. Example: $f(x) = -g(x) = \begin{cases} 1 & \text{for } x \ge 0 \\ -1 & \text{for } x < 0 \end{cases}$ both functions are discontinuous at the point x = 0, their product f(x) g(x) ≡ – 1 being continuous everywhere.

6. No

7. No. Example: $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$

8. $f(x) = \begin{cases} x+1, & x \ge 0 \\ x-1, & x < 0 \end{cases}$

PRACTICE PROBLEMS—D

11. x^2 if x ≥ 0; 0 if x < 0; h is continuous everywhere.

12. gof(x) is discontinous at x = – 1, 0, 1

13. 1 if $1 \le |x| \le \sqrt{3}$; 0 otherwise. h is continuous everywhere except at $x = \pm 1, \pm\sqrt{3}$.

14. discontinuous at 1 and –1.

15. {0, 1}

16. $g(x) = \begin{cases} x+2, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 4-x, & 2 < x \le 3 \end{cases}$ and g(x) is discontinuous at x = 1 and x = 2

17. continuous everywhere

18. h(x) is discontinuous at x = 0

CONCEPT PROBLEMS—E

1. (i) $f(x) = \frac{1}{x-1}$; g(x) = x

(ii) f(x) = x ; $g(x) = \frac{1}{x^2+1}$

(iii) $f(x) = x^2$; $g(x) = \tan^{-1}x$

5. (i) (–4, –3), (–2, –1), (–1, 0), (0, 1),

(ii) (–3, –2.5), (–2.5, –2), (0, 0.5), (0.5, 1),

PRACTICE PROBLEMS—E

8. (a) yes (b) yes

14. (ii) $f(x) = x^2(\sin^2 x + 2)$

CONCEPT PROBLEMS—F

2. (i) c = 3 (ii) c = 2

(iii) c = 3

5. 1.53

10. Yes

14. f must be a constant function.

PRACTICE PROBLEMS—F

24. (– 4, – 3), (0, 1), (4, 5)

26. Yes

30. 0.6 approx.

34. False

37. $\sqrt{5}$

38. (i) [1, 2] (ii) [0, 1]

(iii) $\left[0, \frac{\pi}{2}\right]$ (iv) $\left[-\frac{\pi}{4}, 0\right]$

41. f(x) = 0

OBJECTIVE EXERCISE

1. B	**2.** D	**3.** C
4. C	**5.** D	**6.** B
7. C	**8.** B	**9.** D
10. A	**11.** B	**12.** B
13. C	**14.** A	**15.** D
16. B	**17.** B	**18.** C
19. A	**20.** C	**21.** A
22. C	**23.** C	**24.** A
25. B	**26.** A	**27.** D
28. C	**29.** A	**30.** C
31. B	**32.** A	**33.** A
34. B	**35.** C	**36.** C
37. B	**38.** C	**39.** C
40. C	**41.** B	**42.** C
43. D	**44.** D	**45.** B
46. D	**47.** D	**48.** B
49. B	**50.** C	**51.** D
52. C	**53.** C	**54.** A
55. A	**56.** A	**57.** D
58. C	**59.** C	**60.** C
61. D	**62.** A	**63.** C
64. C	**65.** C	**66.** D
67. A	**68.** C	**69.** B

70. D	**71.** ABCD	**72.** AB
73. ABCD	**74.** AD	**75.** ABC
76. ABCD	**77.** ABCD	**78.** AB
79. BC	**80.** ABCD	**81.** ACD
82. ABC	**83.** ABCD	**84.** AC
85. ABCD	**86.** BCD	**87.** AB
88. CD	**89.** CD	**90.** BCD
91. A	**92.** D	**93.** C
94. B	**95.** C	**96.** D
97. C	**98.** C	**99.** C
100. D	**101.** D	**102.** B
103. B	**104.** B	**105.** D
106. A	**107.** B	**108.** C
109. C	**110.** C	**111.** B
112. C	**113.** D	**114.** A
115. A		

116. (A)–(R); (B)–(PQ); (C)–(QS); (D)–(T)

117. (A)–(RS); (B)–(QR); (C)–(P); (D)–(P)

118. (A)–(R); (B)–(S); (C)–(Q); (D)–(P)

119. (A)–(S); (B)–(R); (C)–(P); (D)–(Q)

120. (A)–(Q); (B)–(R); (C)–(P); (D)–(PS)

REVIEW EXERCISES FOR JEE ADVANDED

1. discontinuous at $x = -1, 0, 1, 2$.

2. continuous in R

3. continuous.

4. discontinuous at $x = 0, \pi$.

5. g is continuous on (1, 2].

6. discontinuous at all positive odd multiples of $\pi/2$

7. 4π

8. discontinuity of the second kind at $x = 0$.

9. (i) discontinuity of the first kind from left at $x = 1, 2, 3, ...$;
(ii) continuous for all $x \geq 0$.

10. $a > 64$

11. $a = e, b = \frac{1}{e} - e$

13. $n = 1$

14. $a = 1 - \frac{1}{e}, b = 0$

17. Discontinuous at 0, 1

18. continuous in domain

19. Discontinuous at all positive rational number except 1, continuous otherwise.

21. discontinuties lie one in each interval : (–2, –1), (0, 1), (1, 2)

23. (fog) x is continuous everywhere and (gof) x is discontinuous at $x = 0, \pm 1$.

30. discontinuous at $x = 0$.

35. 1

TARGET EXERCISES FOR JEE ADVANCED

2. discontinuous

3. continuous everywhere.

4. discontinuous at $x = 0, 1$.

7. $y_n(x)$ is continuous at $x = 0$ for all n and y(x) is discontinuous at $x = 0$

8. $f(x) = 0$ when $x = 0$,
$= \frac{1}{2} - x$ when $0 < x < \frac{1}{2}$,
$= \frac{1}{2}$ when $x = \frac{1}{2}$,
$= \frac{3}{2} - x$ when $\frac{1}{2} < x < 1$,
$= 1$ when $x = 1$,

13. 4

14. $f(x) > 0$ for $c < x \leq b$, and $f(x) < 0$ for $a \leq x < c$.

17. f is continuous at every irrational in $(0, \infty)$ and discontinuous at every rational in $(0, \infty)$

21. (iii) No (iv) yes

22. finite discontinuity at $x = n + 1/2, n \in I$.

23. both are discontinuous; $\phi(x)$ is continuous if $a = 1$.

24. continuous everywhere.

27. f is discontinuous for all finite decimal representations of x in which the last digit is 8 and none of the other digits is 7. For all other values f is continuous.

28. Jump discontinuity.

30. $k = 0; g(x) = \begin{cases} \ell n(\tan x) & \text{if } 0 < x < \frac{\pi}{4} \\ 0 & \text{if } \frac{\pi}{4} \leq x < \frac{\pi}{2} \end{cases}$.

Hence g(x) is continuous everywhere.

31. (i) $4-3\sqrt{2}+a$, (ii) $a=3$

32. $f(0^+)=\frac{\pi}{2}$; $f(0^-)=\frac{\pi}{4\sqrt{2}} \Rightarrow$ f is discontinuous at x = 0 ; $g(0^+)=g(0^-)=g(0)=\pi/2 \Rightarrow$ g is cont. at x = 0.

33. discontinuity of the first kind at 1/r, r = 2, 3, ...; discontinuity of the first kind from left at x = 1; discontinuity of the first kind from right at x = 0.

34. $n+1/2, n \in I$.

PREVIOUS YEAR'S QUESTIONS (JEE ADVANED FOR ADVANCED)

1. $k=7$

2. $f(x)=\sqrt{4-x^2}$, $-2\le x\le 0 = -\sqrt{4-x^2}$, $0\le x\le 2$

3. $(-\infty,-1)\cup[0,\infty)$, $I-\{0\}$ where I is the set of integer except $n=-1$

4. 10 **5.** B

6. C **7.** D

8. CD **9.** BCD

10. A

11. B, D

12. $a=-\frac{3}{2}$, $b\in R$, $c=\frac{1}{2}$

13. $g(x)=\begin{cases}2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 4-x. & 2x<x\le 3\end{cases}$

discontinuity at x = 1, 2

14. $a=\frac{\pi}{6}$, $b=-\frac{\pi}{12}$

15. $a=8$

16. $a=\frac{2}{3}$, $b=e^{2/3}$